AN OPTIMIZATION APPROACH TO DESIGN OF GEOMETRY AND FORCES OF TENSEGRITIES

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Background –1

- Difficulties of shape design of tensegrities
  - Interaction of shape and force
  - Cable (tension) and strut (compression)
- Force density method
  - Cannot specify force and direction

Background –2

- Direct assignment of member directions:
  - Ohsaki and Kanno (IASS-APCS 2003)
  - Variables: member force vector
    nodal coordinates
  - Too many variables to be specified

Objective

- Two stage approach for form finding
  - Step 1: Find member forces
  - Step 2: Find nodal locations
- Direct assignment of direction and force of member.
- Optimization for determination of member force vectors

Equilibrium equations

- Force vector of member k

$$\mathbf{v}_k = (v_k^x, v_k^y, v_k^z)^{\mathrm{T}}$$

- Member force vector

$$\mathbf{v} = (\mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}}, \cdots, \mathbf{v}_m^{\mathrm{T}})^{\mathrm{T}}$$

  – $\mathbf{v}$ has $3m$ components

- Equilibrium equation

$$\mathbf{Bv} = \mathbf{0}$$

Geometrical constraints w.r.t. force vector

- Rotational symmetry

$$\mathbf{v}_{k+1} = \mathbf{M}_k \mathbf{v}_k \qquad \mathbf{M}_k = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Direct constraints $\mathbf{S}\mathbf{v} = \mathbf{0}$ $\qquad \mathbf{v}_1 = \mathbf{v}_2$

Geometrical constraints w.r.t. force vector

- Geometrical constraints and equilibrium
  - Hard constraints

$$\mathbf{C}\mathbf{v} = \mathbf{0} \qquad \mathbf{C} = \left[\mathbf{B}^{\mathrm{T}}, \mathbf{M}^{\mathrm{T}}, \mathbf{S}^{\mathrm{T}}\right]^{\mathrm{T}}$$

- r : rank of **C**
- Specfy 2m-r components of **v** to obtain **v**

$\Rightarrow$ Not straightforward

Objective functions and constraints

- Soft constraints $\mathbf{Rv} = \mathbf{0}$

  e.g. $\mathbf{v}_7 = c\mathbf{v}_9$

- Objective function

$$E(\mathbf{v}) = \frac{1}{2}\left(\mathbf{v} - \overline{\mathbf{v}}\right)^{\mathrm{T}} \mathbf{W}^{\mathrm{I}} \left(\mathbf{v} - \overline{\mathbf{v}}\right) + \frac{1}{2}(\mathbf{Rv})^{\mathrm{T}} \mathbf{W}^{\mathrm{II}} (\mathbf{Rv})$$

- Constraints (hard constraints)

$$\mathbf{Cv} = \mathbf{0}$$

Optimization problem

- Minimize $E(\mathbf{v})$
- subject to $\mathbf{C}\mathbf{v} = \mathbf{0}$

- Lagrangian

$$L(\mathbf{v}, \boldsymbol{\mu}) = E(\mathbf{v}) + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}\mathbf{v}$$

- Stationary condition

$$\begin{bmatrix} \mathbf{W}^{\mathrm{I}} + \mathbf{R}^{\mathrm{T}}\mathbf{W}^{\mathrm{II}}\mathbf{R} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{O} \end{bmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{W}^{\mathrm{I}}\overline{\mathbf{v}} \\ \mathbf{0} \end{pmatrix}$$

Equilibrium w.r.t. nodal coordinates

- Direction vector $\mathbf{d}_k = (d_k^x, d_k^y, d_k^z)^{\mathrm{T}}$
- Express **d** by **X**

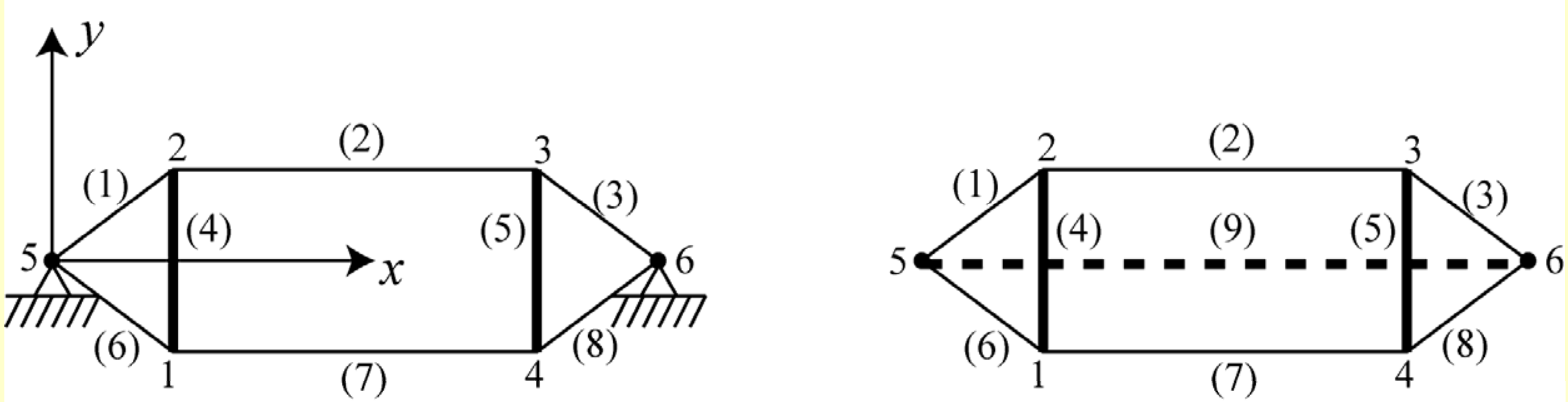
- Equilibrium equation

$$\mathbf{FX} = \mathbf{0}$$

Form finding algorithm

- **Step 0:** Specify topology.
- **Step 1:** Construct the equilibrium matrix and specify the geometrical constraints (hard constraints).
- **Step 2:** Assign the target force vector , the soft constraints, to define the objective function.
- **Step 3:** Solve stationary condition for force vector.
- **Step 4:** Formulate the equilibrium equation with respect to the nodal coordinates.
- **Step 5:** Compute the rank of and specify independent components of to obtain nodal coordinates.

Numerical example



$$E(\mathbf{v}) = \frac{1}{2}(v_5^x - 0.5)^2 + \frac{1}{2}(v_5^y - 1)^2 + \frac{1}{2}(v_9^x - (-3))^2 + \frac{1}{2}(v_9^y - 0)^2 + \frac{1}{2}(v_1^x - (-v_3^x))^2 + \frac{1}{2}(v_1^x - v_3^y)^2$$

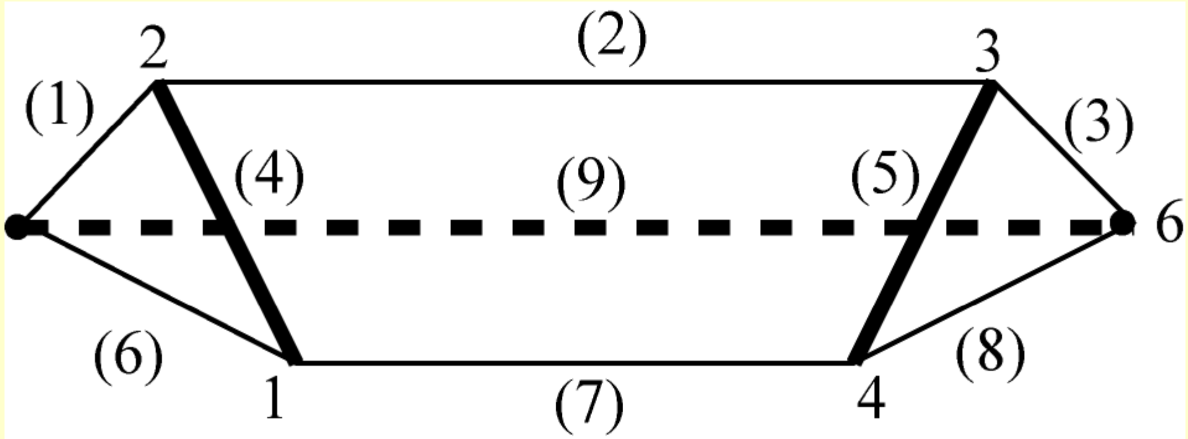
$$+ \frac{1}{2}(v_2^x - v_7^x)^2 + \frac{1}{2}(v_2^y - v_7^y)^2 + \frac{1}{2}(v_6^x - (-v_8^x))^2 + \frac{1}{2}(v_6^y - v_8^y)^2$$

Table 1. Force vectors of the plane tensegrity

Member	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
v_i^x	–1	1.5	1	0.5	0.5	–2	1.5	2	3
v_i^y	–1	0	–1	–1	1	1	0	1	0

Table 2. Nodal coordinates

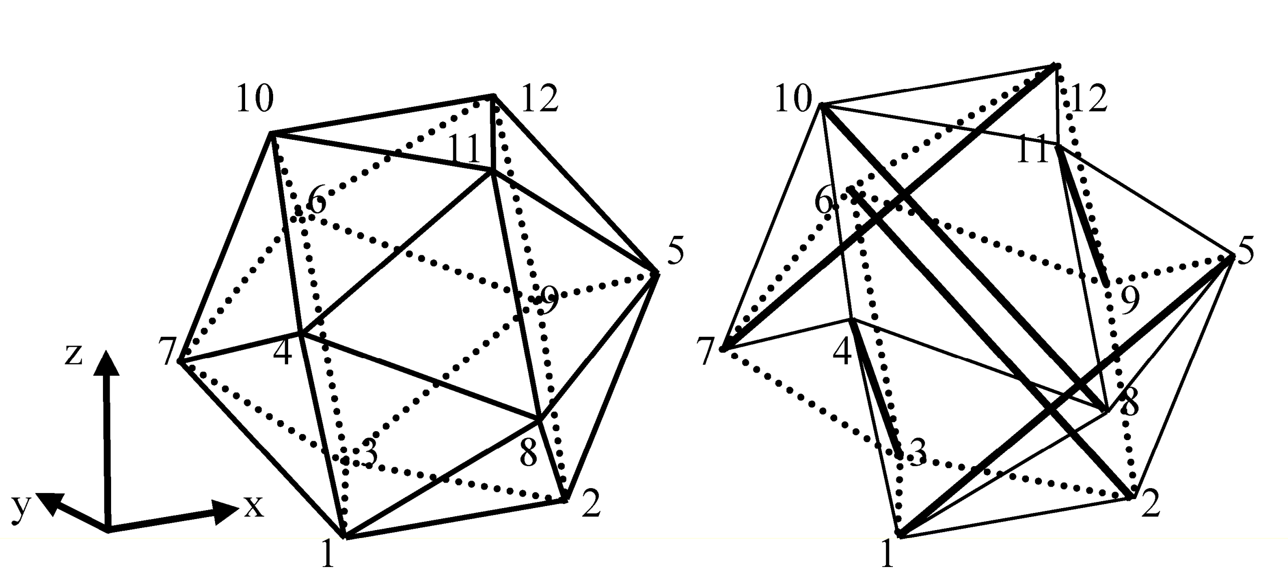
Node	1	2	3	4	5	6
x	2	1	7	6	0	8
y	–1	1	1	–1	0	0


(2)
2
3
(1)
(3)
(4)
(9)
(5)
6
(6)
(8)
1
(7)
4

3D-structure

$$\mathrm{rank}(\mathbf{F}) = 8 \qquad n = 12$$

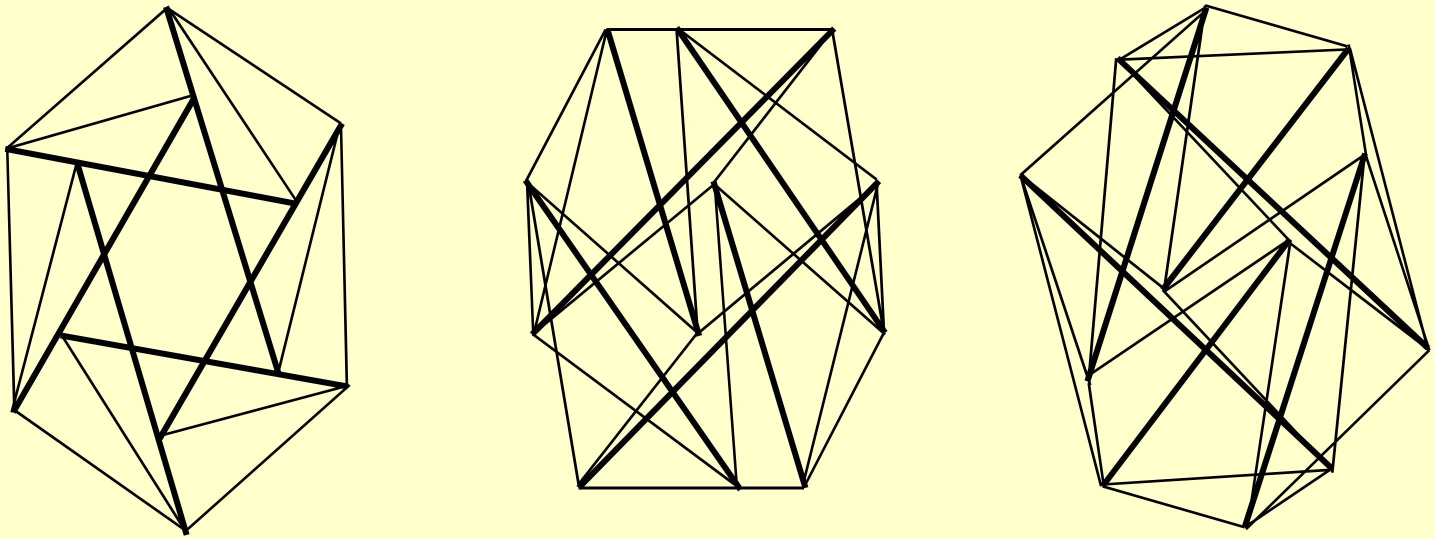
$$(x_2, x_5, y_5, x_6) = (1, 0, 0, 8)$$



3D-structure (Example 1)

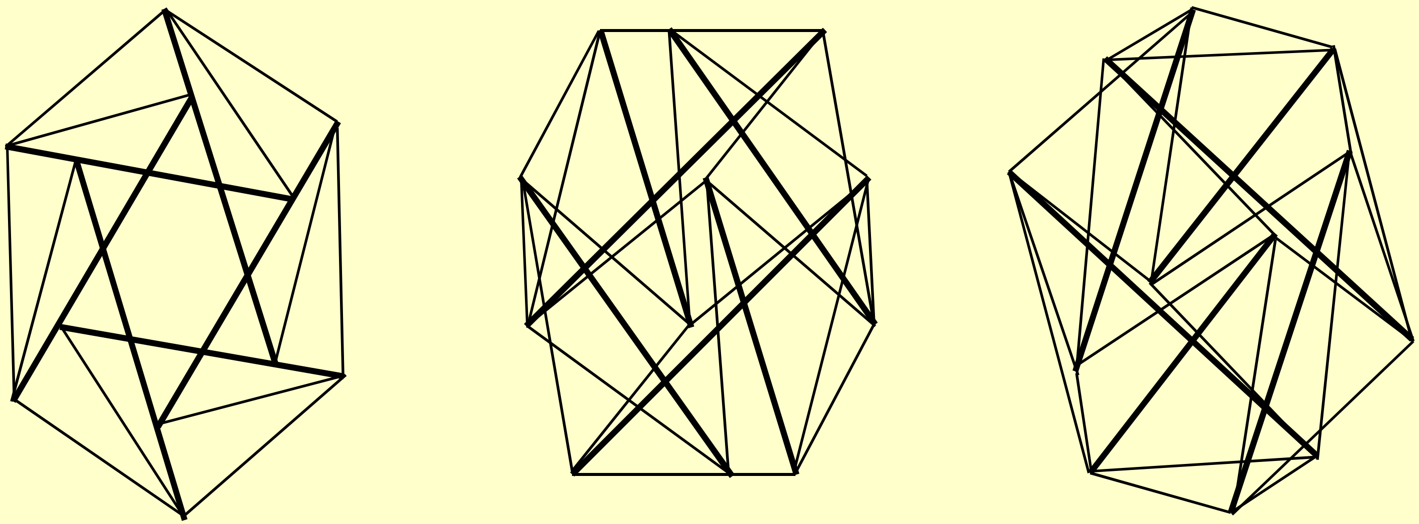
$\mathrm{rank}(\mathbf{F}) = 8$ $\quad n = 12$

$(x_2, x_5, y_5, x_6) = (1, 0, 0, 8)$



3D-structure (Example 2)

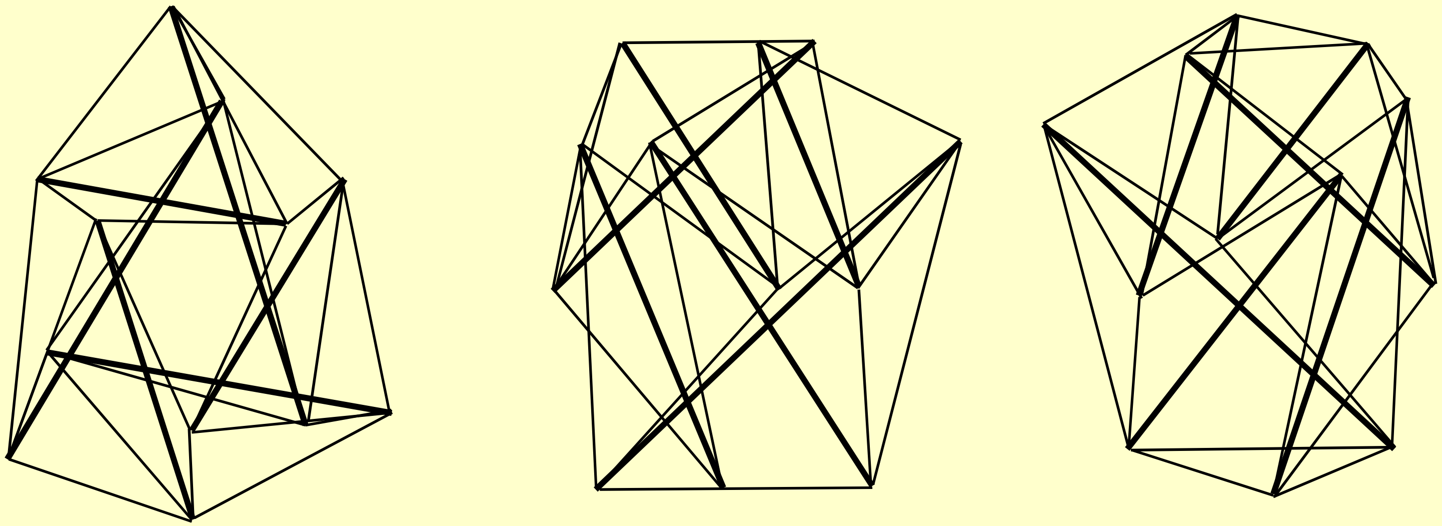
Add soft constraint $\mathbf{v}_7 = c\mathbf{v}_{19}$



result $|\mathbf{v}_7| \cong 1.7|\mathbf{v}_{19}|$

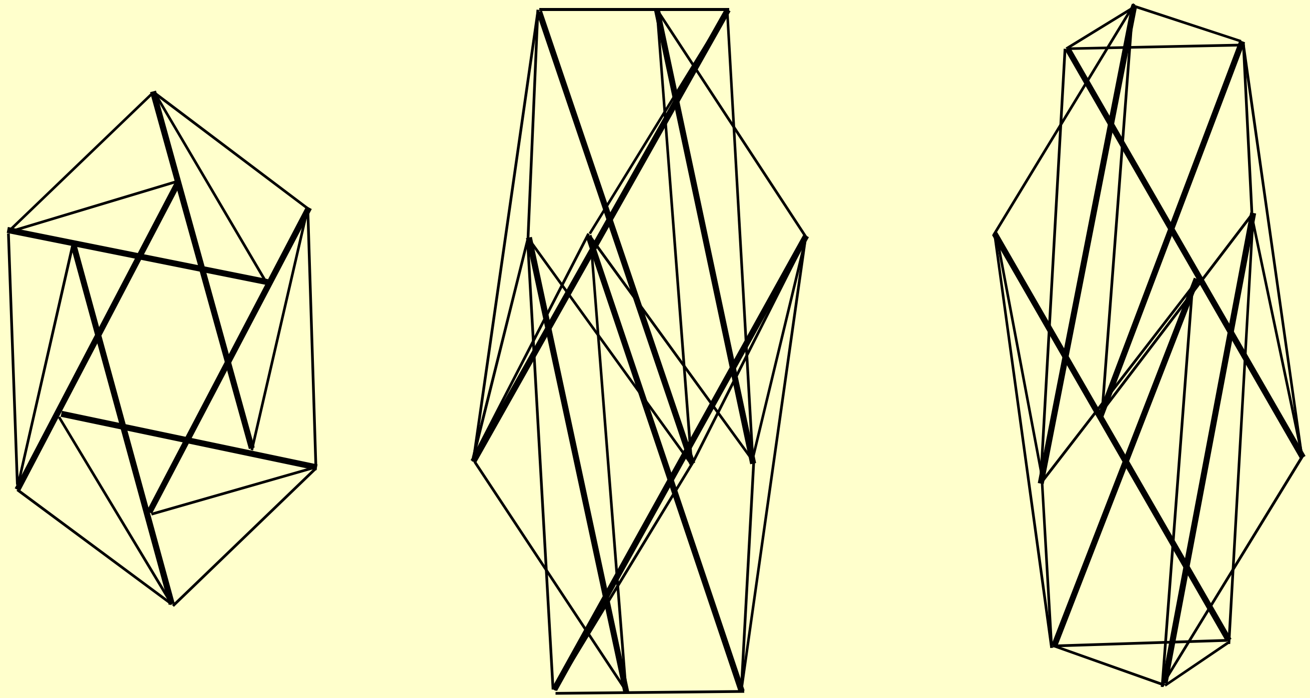
3D-structure (Example 3)

$$E_3(\mathbf{v}) = \frac{1}{2}\sum_{i\in K}(v_i - \overline{v}_i)^2 + \frac{1}{2}(v_7^x - 0.75)^2 + \frac{1}{2}(v_7^y - (-1.83))^2 + \frac{1}{2}(v_7^z - (-4.20))^2$$
$$+ \frac{1}{2}(v_{19}^x - 0.75)^2 + \frac{1}{2}(v_{19}^y - (-1.83))^2 + \frac{1}{2}(v_{19}^z - (-4.20))^2$$



3D-structure (Example 4)

$$E_4(\mathbf{v}) = \frac{1}{2}\sum_{i\in K}(\mathbf{v}_i - \mathbf{v}_i^{\mathrm{Ex3}})^{\mathrm{T}}(\mathbf{v}_i - \mathbf{v}_i^{\mathrm{Ex3}}) + \frac{1}{2}(v_7^x - 2v_{19}^x)^2 + \frac{1}{2}(v_7^y - 2v_{19}^y)^2 + \frac{1}{2}(v_7^z - 2v_{19}^z)^2$$



Conclusions

- Direct assignment or force vectors.
- Member direction can be specified.
- Hard constraints that should be satisfied.
- Soft constraints that are preferably satisfied.
- Determine force components by optimization.
- Shape and forces can be directly controlled as expected by modifying the target values and soft constraints on force components.