Multiobjective Shape Optimization of Latticed Shells for Elastic Stiffness and Uniform Member Lengths

Makoto Ohsaki (Hiroshima University)
Shinnnosuke Fujita (Kanebako Struct. Eng.)
Background:
Optimization of Shell Roofs

Structural performance → Minimum strain energy

+ 

Constraints

Cost performance → Performance measures: weight, volume, etc.
(Formulation is straightforward)

Aesthetic aspect 
Roundness, convexity, planeness, etc.

Constructability
Developable surface → Reduce cost for scaffolding

Algebraic invariants of tensor algebra for differential geometry
Shape representation using Bezier surface

Tensor product Bezier surface

\[ S_{I,J}(s, t) = (x(s, t), y(s, t), z(s, t))^\top = \sum_{i=0}^{I} \sum_{j=0}^{J} q_{ij} B_{I,i}(s) B_{J,j}(t) \]

\( B \): Bernstein basis function \( q_{ij} = (q_{x,i,j}, q_{y,i,j}, q_{z,i,j})^\top \)

\[ S_{3,3}(s, t) = \sum_{i=0}^{3} \sum_{j=0}^{3} q_{ij} B_{3,i}(s) B_{3,j}(t) \]

Tensor product Bezier surface of order 3
Shape representation using Bezier surface

**Triangular patch Bézier surface**

\[ S_n(u, v) = [x(u, v), y(u, v), z(u, v)]^\top = \sum_{i=0}^{n} \sum_{j=0}^{n-i} q_{ij} B_{n,ij}(u, v) \quad u, v \in [0, 1] \]

\[ B_{n,ij}(u, v) = \frac{n!}{i!j!(n-i-j)!} u^i v^j (1 - u - v)^{n-i-j} \]

\[ q_{ij} = (q_{x,ij}, q_{y,ij}, q_{z,ij})^\top \]

\[ S_4(u, v) = \sum_{i=0}^{4} \sum_{j=0}^{4-i} q_{ij} B_{4,ij}(u, v) \]

ex). Triangular Bézier patch of order 4
Definition of algebraic invariants of differential geometry

Covariant component: subscript, underbar
Contravariant component: superscript, overbar

Gradient

$$z_s = \frac{\partial z}{\partial s} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{z,ij} \frac{\partial B_{n,ij}}{\partial s} \quad s \in \{u, v\}$$

$$z = \begin{bmatrix} z_u \\ z_v \end{bmatrix}$$

Covariant Hessian

$$h_{st} = \frac{\partial^2 z}{\partial s \partial t} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{z,ij} \frac{\partial^2 B_{n,ij}}{\partial s \partial t} \quad s, t \in \{u, v\}$$

$$h = \begin{bmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{bmatrix}$$

Covariant metric tensor

$$g_{st} = \frac{\partial S_n}{\partial s}^\top \frac{\partial S_n}{\partial t} \quad s, t \in \{u, v\}$$

$$g = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix}$$

$$\frac{\partial S_n}{\partial s} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{bmatrix}^\top$$
Definitions of algebraic invariants of differential geometry

\begin{align*}
\beta_0 &= \sum_{\xi=s,t} \sum_{\lambda=s,t} g^{\xi\lambda} z_\xi z_\lambda = \sum_{\xi=s,t} z^\xi z_\xi \ (\geq 0) \\
\beta_1 &= \sum_{\xi=s,t} \sum_{\lambda=s,t} h_{\lambda\xi} g^{\xi\lambda} \\
\beta_2 &= \frac{1}{2\text{det}(g)} \sum_{\xi=s,t} \sum_{\lambda=s,t} \sum_{\mu=s,t} \sum_{\nu=s,t} h_{\nu\lambda} h_{\mu\xi} \tilde{E}^{\xi\lambda} \tilde{E}^{\mu\nu}
\end{align*}

\begin{align*}
\gamma_1 &= \sum_{\lambda=s,t} \sum_{\xi=s,t} h_{\lambda\xi} z^\xi z_\lambda \\
\gamma_2 &= \sum_{\lambda=s,t} \sum_{\xi=s,t} h_{\lambda\xi} \tilde{z}^\xi z_\lambda = \sum_{\lambda=s,t} \sum_{\xi=s,t} h_{\lambda\xi} z^\xi \tilde{z}_\lambda \\
\gamma_3 &= \frac{1}{\text{det}(g)} \sum_{\lambda=s,t} \sum_{\xi=s,t} h_{\lambda\xi} \tilde{z}^\xi \tilde{z}_\lambda \\
\alpha &= \frac{1}{4} (\kappa_1 - \kappa_2)^2 = \frac{1}{4} \beta_1^2 - \beta_2
\end{align*}

- \(\beta_1\): twice the mean curvature
- \(\beta_2\): the Gaussian curvature
- \(\gamma_1/\beta_0\): the curvature in the steepest descent direction
- \(\gamma_3/\beta_0\): the curvature in the direction perpendicular to the steepest descent direction
- \(\alpha\): roundness measure
Relation between the invariants and surface shape

Local properties in the neighborhood of a point P on the surface are characterized by the invariants as follows:

<table>
<thead>
<tr>
<th>Invariant Condition</th>
<th>Property Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_2 &gt; 0$</td>
<td>The contours in the neighbourhood of P are coaxial (part of) similar ellipses.</td>
<td></td>
</tr>
<tr>
<td>$\beta_1 &gt; 0$</td>
<td>The shape is locally concave. (example : ●)</td>
<td></td>
</tr>
<tr>
<td>$\beta_1 &lt; 0$</td>
<td>The shape is locally convex. (example : ●)</td>
<td></td>
</tr>
<tr>
<td>$\beta_2 &lt; 0$</td>
<td>The contours in the neighbourhood of P are (part of) coaxial hyperbolas. (example : ○)</td>
<td></td>
</tr>
<tr>
<td>$\beta_0 = 0$</td>
<td>P is a critical point. (example : ●)</td>
<td></td>
</tr>
</tbody>
</table>

![Diagram](image)
Relation between the invariants and surface shape

\[ \beta_2 = 0 \]

One of the principal curvatures is 0.

\[ \beta_1 > 0 \rightarrow \text{The other principal curvature is positive. (example : ● )} \]

\[ \beta_1 < 0 \rightarrow \text{The other principal curvature is negative. (example : ● )} \]

\[ \beta_1 = 0 \rightarrow \text{The shape is locally flat surface. (example : ● )} \]
Relation between the invariants and surface shape

\[ \gamma_2 = 0 \]

Direction of gradient vector coincides with one of the principal direction, and the surface near P is locally cylindrical.

\(|\gamma_1| < |\gamma_3|\) and \(\gamma_2 > 0\) \(\rightarrow\) concave in one principal direction (example: \(\bullet\))

\(|\gamma_1| > |\gamma_3|\) and \(\gamma_2 < 0\) \(\rightarrow\) convex in one principal direction (example: \(\bullet\))
Locally convex surface

minimize \( f(q_z) = \frac{1}{2} d^\top K d \)

subject to

\[
\begin{align*}
S - S_0 &\leq 0 \\
r_z^* - r_{z,0}^* &\leq 0 \\
\beta_{2}^c &> 0 \\
\beta_{1}^c &\leq \bar{\beta}
\end{align*}
\]

Constraints \( \bar{\beta} < 0 \)
to obtain a locally convex surface.

Two cases, \( \bar{\beta} = -0.1 \) and \( \bar{\beta} = -0.15 \)
More convexity for larger absolute value of \( \bar{\beta} \)
Locally convex surface with large stiffness
Locally cylindrical surface

minimize \( f(q_z) = \frac{1}{2} d^\top K d \)

subject to

\[
\begin{align*}
S - S_0 &\leq 0 \\
\gamma_{c1} - \gamma_{c2} &\leq 0 \\
\gamma_{c1} - \gamma_{c2} &\leq 0 \\
\gamma_{c1} - \gamma_{c2} &\leq 0 \\
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\gamma_{c1} - \gamma_{c2} &\leq 0 \\
\gamma_{c1} - \gamma_{c2} &\leq 0 \\
\gamma_{c1} - \gamma_{c2} &\leq 0
\end{align*}
\]

Constraint points

\( (s^{c1}, t^{c1}) = (0.25, 0.25) \)

\( (s^{c2}, t^{c2}) = (0.75, 0.75) \)

Constraints \( \bar{\gamma}^{c1}, \bar{\gamma}^{c2} < 0 \)

to obtain locally cylindrical and convex surface.

Two cases, \( \bar{\gamma}^{c1} = \bar{\gamma}^{c2} = -0.015 \) and \( -0.025 \)

More cylindricity and convexity for larger absolute value
Initial shape

Max. compressive stress : 7.1183
Max. tensile stress : 3.0838
Max. bending stress : 7.9380

Optimum shape (c1 = c2 = -0.015)

Max. compressive stress : 3.2601
Max. tensile stress : 0.3743
Max. bending stress : 1.3615

Optimum shape (c1 = c2 = -0.025)

Max. compressive stress : 3.1842
Max. tensile stress : 0.7586
Max. bending stress : 1.0645

Contour line

Locally cylindrical surface with large stiffness.
Multiobjective programming for roundness and stiffness

Optimization problem

\[
\begin{align*}
\text{minimize} \quad & f(q_z) = \frac{1}{2} d^\top K d \\
\text{subject to} \quad & g(q_z) = \sum_{i=1}^{15} \alpha^c_i \\
& S - S_0 \leq 0 \\
& r_z^* - r_{z,0}^* = 0
\end{align*}
\]

Constraint approach

Minimize strain energy

\[
\begin{align*}
\text{minimize} \quad & f(q_z) \\
\text{subject to} \quad & S - S_0 \leq 0 \\
& r_z^* - r_{z,0}^* = 0 \\
& g(q_z) - \bar{g} \leq 0
\end{align*}
\]

Maximize roundness

\[
\begin{align*}
\text{minimize} \quad & g(q_z) \\
\text{subject to} \quad & S - S_0 \leq 0 \\
& r_z^* - r_{z,0}^* = 0 \\
& f(q_z) - \bar{f} \leq 0
\end{align*}
\]

Find optimal solutions for different values of $\bar{f}$ and $\bar{g}$
Multiobjective programming for roundness and stiffness
Developable surface

\[ \min f(q) = \frac{1}{2} d^\top K d \]

subject to
\[
\begin{align*}
S - S_0 &\leq 0 \\
\beta_2^{ci} &\equiv 0 \\
(i=1,\ldots,25)
\end{align*}
\]

\( \beta_2 \) vanishes at 25 points indicated by the dots in the figure, generating a developable surface.
Developable shape
Optimization of Latticed Shell

**Structural Performance**
- Strain energy
- Compliance

minimize

+ Optimal shape with large stiffness

**Non-structural Performance**
- Geometrical property
- Constructability
- Uniform member length
- Minimum number of different joints
Performance measures

Strain energy: 
\[ f = \frac{1}{2} d^\top K d \]

Variance of member length: 
\[ g = \sum_{k=1}^{mn} (l_k - l_{\text{ave}})^2 \]

Constraint on total member length: 
\[ L - L_0 = 0 \]

- \( d \): nodal displacement vector
- \( K \): stiffness matrix
- \( mn \): number of members
- \( l_k \): length of \( k \)th member
- \( l_{\text{ave}} \): average ember length
- \( L \): total member length
Optimization Problem

Multiobjective Optimization

Minimize $f(x)$ and $g(x)$
subject to $L(x) = L_0$

$f(x)$: strain energy
$g(x)$: variance of member length

Constraint Approach

Minimize $f(x)$
subject to $g(x) = 0$
$L(x) = L_0$

Minimize $g(x)$
subject to $f(x) = f$
$L(x) = L_0$
Triangular grids

Design variables:
Locations of control points

$q_x, q_y, q_z$

Frame model
Triangular Bezier patch
\[
\begin{align*}
q_x &= [q_{x2}, q_{x3}, \ldots, q_{x6}, q_{x8}, q_{x9}, \ldots, q_{x27}]^T \\
q_y &= [q_{y2}, q_{y3}, \ldots, q_{y6}, q_{y8}, q_{y9}, \ldots, q_{y27}]^T \\
q_z &= [q_{z2}, q_{z3}, \ldots, q_{z6}, q_{z8}, q_{z9}, \ldots, q_{z27}]^T
\end{align*}
\]

\[
\begin{align*}
\minimize & \quad f(q_x, q_y, q_z) \\
\text{subject to} & \quad \{ \begin{array}{l}
L(q_x, q_y, q_z) - L_0 = 0 \quad \text{total length}
\end{array} \}
\]

\[f(q_x, q_y, q_z) = 0 \quad \text{uniform member length}
\]

No feasible solution
Initial $\bar{f} = 1.3$  
$\bar{f} = 2.0$  
$\bar{f} = 0.5$  
$\bar{f} = 0.05$

Uniform member length $\Rightarrow$ small stiffness

Small strain energy $\Rightarrow$ unrealistic shape

Allow small deviation of member length $\Rightarrow$ stiff and realistic shape

$\max - \min$

$1054\text{mm}$

$5.165\text{mm}$

$58.24\text{mm}$

$3045\text{mm}$

$0.363\text{mm}$

$20.42\text{mm}$

$12.52\text{mm}$

$3.324\text{mm}$
minimize \[ f(q_x, q_y, q_z) \]
subject to \[ \begin{cases} L(q_x, q_y, q_z) - L_0 = 0 \\ g(q_x, q_y, q_z) = 0 \end{cases} \]

Uniform member length
Cylindrical shape
Small stiffness
Quadrilateral grid

Frame model

Tensor product Bezier surface
Fixed support

Fixed point
Initial

Large strain energy
Small deviation of member length

\[ f = 8.226 \]
\[ l_{\text{max}} - l_{\text{min}} = 422.2 \]

Small strain energy
Large deviation of member length

\[ f = 1.270 \]
\[ l_{\text{max}} - l_{\text{min}} = 0.07277 \]

\[ f = 0.4162 \]
\[ l_{\text{max}} - l_{\text{min}} = 0.9248 \]
Initial $\bar{f} = 0.032$

- $f = 0.08$
  - $\bar{f} = 0.040$
  - $\bar{f} = 0.005$

Almost uniform member length

$\ell_{\text{max}} - \ell_{\text{min}} = 422.2\,\text{mm}$

$\delta_{\text{max}} = 0.840\,\text{mm}$

- $3.340\,\text{mm}$
- $3.792\,\text{mm}$
- $8.418\,\text{mm}$
- $1.665\,\text{mm}$
- $93.59\,\text{mm}$
- $0.144\,\text{mm}$
Pareto optimal solutions from different initial solutions
Hexagonal Grid

Do not use parametric surfaces
Use symmetry conditions
minimize \quad f(q_x, q_y, q_z) \\
subject to \quad \left\{ \begin{array}{c}
L(q_x, q_y, q_z) - L_0 = 0 \\
g(q_x, q_y, q_z) = 0
\end{array} \right.
$f = 2.750 \quad l_{\text{max}} - l_{\text{min}} = 536.2$

Optimal shape: case 1

$f = 0.08689 \quad l_{\text{max}} - l_{\text{min}} = 0.002850$
Optimal shape: case 2

\[ f = 0.02636 \quad l_{\text{max}} - l_{\text{min}} = 0.002579 \]

Optimal shape: case 3

\[ f = 0.05488 \quad l_{\text{max}} - l_{\text{min}} = 0.005229 \]
Conclusions

• Multiobjective shape optimization of latticed shells.
  – Objective functions: strain energy and variance of member lengths.
  – Optimal shapes for triangular, quadrilateral, and hexagonal grids.
  – Constraint approach for converting the multiobjective problem to a single objective problem.
• Feasible solution with uniform member lengths
  ⇒ Minimize strain energy under uniform member lengths.
• No feasible solution
  ⇒ Minimize variance of member length for specified strain energy.
Conclusions

- Optimal shape of triangular grid with uniform member lengths
  ⇒ cylindrical surface with equilateral triangles.

- Optimal shapes of quadrilateral grid
  ⇒ highly dependent on initial solution;
    bifurcation in objective function space.

- Various shapes with uniform member lengths
  for hexagonal grids.