Design of deployable structures using limit analysis of partially rigid frames with quadratic yield functions

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Planar mechanism (deployable structure) with partially rigid joints

Generate mechanism with partially rigid joints using optimization method

- ●: rotational hinge
- -- --: removed member

Input disp. ↔ Output disp.
Mechanism with partially rigid joints

• Optimization approaches for generating link mechanisms
  – Truss elements for planar mechanism
  – Ideal (three-axis) pin joints are needed for 3D mechanism

• Ideal pin joints are difficult to manufacture
  → partially rigid connections are preferred

• No systematic approach to design of 3D link mechanism with partially rigid connections
Definition of local coordinates and member-end force

Local coordinates

<table>
<thead>
<tr>
<th>Release moment</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Torsion around $x$-axis</td>
</tr>
<tr>
<td>$My$</td>
<td>Bending around $y$-axis</td>
</tr>
<tr>
<td>$Mz$</td>
<td>Bending around $z$-axis</td>
</tr>
</tbody>
</table>

Rotational hinge around $y$-axis
Definition of local coordinates and member-end force

6 x 2 member-end forces
Six equilibrium equations
→ Six independent components

\[ f^k = (N^k, T^k, M^k_{yi}, M^k_{zi}, M^k_{yj}, M^k_{zj})^T \]
Limit analysis problem with two loading conditions

\[
\begin{align*}
\text{max. } & \quad \lambda_{\text{in}} \\
\text{s. t. } & \quad \sum_{i=1}^{n} f_i h_i = \lambda_{\text{in}} p_{\text{in}} + p_{\text{out}} \\
& \quad \alpha w_i \geq |f_i|, \quad (i = 1, \ldots, n)
\end{align*}
\]

\(\lambda_{\text{in}}\) : load factor  \\
\(h_i\) : \(i\)th row of equilibrium matrix  \\
\(p_{\text{in}}\) : load vector at input node  \\
\(p_{\text{out}}\) : load vector at output node  \\
\(\alpha w_i\) : yield moment or yield member force
Procedure for generating infinitesimal mechanism

• **Step 1**: Define equilibrium matrix of rigidly jointed frame.

• **Step 2**: Find release conditions solving limit analysis problem.

\[ \alpha w_i = |f_i| \quad \text{for bending moment} \rightarrow \text{rotational hinge} \]
\[ \alpha w_i = |f_i| \quad \text{for torsional moment} \rightarrow \text{torsional hinge} \]
\[ \alpha w_i = |f_i| \quad \text{for axial force} \rightarrow \text{remove member} \]

• **Step 3**: Output nodal displacements (mechanism) given as dual variables.
Procedure for generating finite mechanism

• **Step 1**: Find release conditions solving limit analysis problem.

• **Step 2**: Geometrical non-linear analysis.
  – If there exist internal forces, release the corresponding member-end force and continue this process.

• **Step 3**: Output the release conditions and nodal displacements of mechanism.
Planar model 1

unit size: $1 \times 1$

$\bar{u}_{in} = 0.3$

$w_i = 1.0$ for member extension

$w_i = 0.0001$ for hinge rotation
Planar model 1

Global mechanism

Local mechanism

\[ 0.42 \leq \alpha \leq 0.63 \]

\( \alpha \geq 0.64 \)

(\( \alpha \leq 0.41 \rightarrow \) objective function is not bounded below)

● : rotational hinge

--- : removed member
Planar model 2

unit size: $1 \times 1$

$\bar{u}_{in} = 0.3$

$w_i = 1.0$ for member extension

$w_i = 0.0001$ for hinge rotation
Planar model 2

Local mechanism

Global mechanism

\[ \alpha \geq 0.42 \]

\[ \alpha \geq 0.42, \quad w_i = 10000.0 \text{ for extension of member A} \]

\[ \alpha \leq 0.41 \rightarrow \text{objective function is not bounded below} \]

● : rotational hinge

- - - - : removed member
3D mechanism of grid model

Symmetric w.r.t. XZ- and YZ-planes
Node 1: fix rotations around X- and Y-axes
Nodes 2 and 4: Fix displacements in Y- and Z-directions
Nodes 3 and 5: Fix displacements in X- and Z-directions
3D mechanism of grid model

- : rotational hinge around local y-axis (bending)
● : rotational hinge around local z-axis (bending)
X : rotational hinge around local x-axis (torsion)
3D mechanism of grid model

Deformation of 3D mechanism without external load
Boundary condition
- fixed in Z-direction
- fixed in X-direction
- fixed in Y-direction

node 1, 2 and 3: fixed around Y-direction
node 4, 11 and 18: fixed around X-direction
3D model 1

Pull node 1 downward
\[ \rightarrow \]
Node 10 moves upward

unit size: $1 \times 1$

$\beta = 1.0$

$w_i = 1.0$ for member extension

$w_i = 0.1$ for bending and torsion
3D model 2
Partially rigid frame with diagonal hinges

Diagonal hinge (arbitrary direction)
→ More diverse deformation
Reduce number of hinges
Optimization problem for mechanism with diagonal hinges

maximize $\lambda_{in}$ 

subject to $\sum_{i=1}^{6m} f_i h_i = p_{out} + \lambda_{in} p_{in}$ 

yield condition for moment

$$(T^k(f))^2 + (M_{j2}^k(f))^2 + (M_{j3}^k(f))^2 \leq \alpha w^b, \quad (k = 1, \ldots, m; j = 1, 2)$$

yield condition for axial force

$$(N^k(f))^2 \leq \alpha w^a, \quad (k = 1, \ldots, m)$$

load factor

equilibrium

quadratic yield condition conditions
Optimality conditions

Normalization of $u$
\[
1 - p_{in}^T u = 0
\]

Bending moment
\[
h_i^T u + 2M_{jp}^k (f) c_j^k = 0,
\]
\[
(k = 1, \ldots, m; \ j = 1, 2; \ p = 2, 3)
\]

Torsional moment
\[
h_i^T u + 2T^k (f)(c_1^k + c_2^k) = 0,
\]
\[
(k = 1, \ldots, m; \ j = 1, 2)
\]

$j$ : member-end
$p$ : axis
$c_j^k$ : rotation of member-end $j$
$M_{jp}^k$ : bending moment at member-end $j$ around axis $p$
$T^k$ : torsional moment
Optimality conditions

Axial force
\[ h_i^T u + 2N^k(f)c_0^k = 0, \]
\[ (k = 1, \ldots, m) \]

Complementarity condition

\[ [(T^k(f))^2 + (M^k_{j2}(f))^2 + (M^k_{j3}(f))^2 - \alpha w^b] c_{kj} = 0, \]
\[ c_{kj} \geq 0, \quad (k = 1, \ldots, m; j = 1, 2) \]

\[ [(N^k(f))^2 - \alpha w^a] c_0^k = 0, \]
\[ c_0^k \geq 0, \quad (k = 1, \ldots, m; j = 1, 2) \]

Rotation or extension is non-zero only when yield condition is satisfied with equality.
Optimality condition

Bending moment

\[ \theta_{jp}^k = 2M_{jp}^k c_j^k, \quad (k = 1, \ldots, m; \quad j = 1, 2; \quad p = 1, 2) \]

Rotation \( \theta_{jp}^k \) around axis \( p \) is proportional to bending moment \( M_{jp}^k \)

(norm of rotation \( c_j^k \) does not depend on axis)

Torsional moment

\[ \theta_1^k = \theta_{j1}^k - \theta_{i1}^k = 2T^k (c_1^k + c_2^k), \quad (k = 1, \ldots, m; \quad j = 1, 2) \]

\[
\begin{align*}
\mathbf{R}_1^k = \begin{pmatrix}
-\theta_1^k \\
\theta_{12}^k \\
\theta_{13}^k
\end{pmatrix} &= c_{k1} \begin{pmatrix}
-T^k \\
M_{12}^k \\
M_{13}^k
\end{pmatrix} \\
\mathbf{R}_2^k = \begin{pmatrix}
\theta_1^k \\
\theta_{21}^k \\
\theta_{23}^k
\end{pmatrix} &= c_{k2} \begin{pmatrix}
T^k \\
M_{21}^k \\
M_{23}^k
\end{pmatrix}
\end{align*}
\]
Auxiliary problem for determination of parameter alpha

maximize \( \mu \) \hspace{1cm} \text{load factor} \\
subject to \( \sum_{i=1}^{6m} f_i h_i = \mu p_{\text{out}} \) \hspace{1cm} \text{equilibrium} \\
\text{yield condition for moment} \\
( (T^k(f))^2 + (M^k_{j2}(f))^2 + (M^k_{j3}(f))^2 \leq w^b, \\
(k = 1, \ldots, m; j = 1, 2) \\
\text{yield condition for axial force} \\
(N^k(f))^2 \leq w^a, \hspace{0.5cm} (k = 1, \ldots, m) \\
\text{lower bound of } \alpha = 1 / \mu
Example 1
Example 1

Vertical members 1-3 and 1-5

Horizontal members 1-2 and 1-4
Example of 3D-mechanism with partially rigid joints

Symmetric w.r.t. XZ- and YZ-planes
Node 1: fix rotations around X- and Y-axes
Nodes 2 and 4: Fix displacements in Y- and Z-directions
Nodes 3 and 5: Fix displacements in X- and Z-directions

Boundary condition

Specified deformation
Example 2
Conclusions

• New method for generating a deployable structure by solving a quadratic programming problem.
• Limit analysis problem with a quadratic yield function of the member-end moments.
• Mechanism with diagonal hinges.
• By allowing a diagonal hinge, the number of hinges can be reduced compared with the previous study, where only the hinges around the local axes are allowed.
• Additional hinges may be needed to generate a finite mechanism.