Discrete elastica for shape design of gridshells

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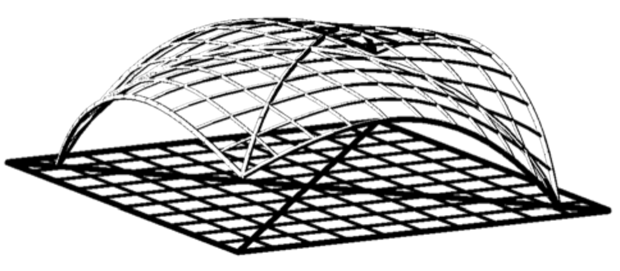
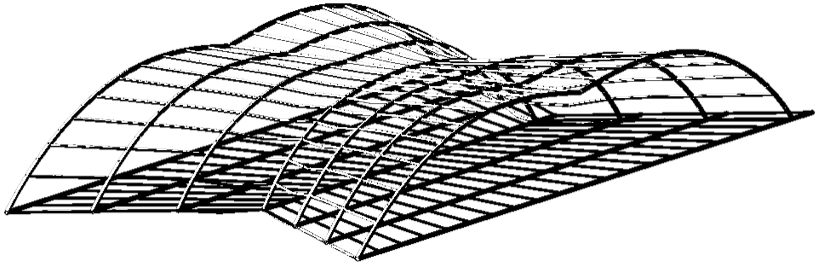
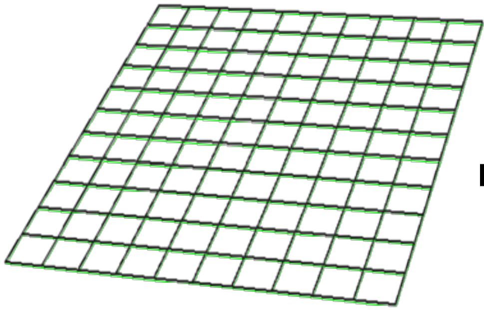




Y. Sakai and M. Ohsaki, Discrete elastica for shape design of gridshells, Eng. Struct., Vol. 169, pp. 55-67, 2018.

Problem of the generating process

Engineer



Initial shape

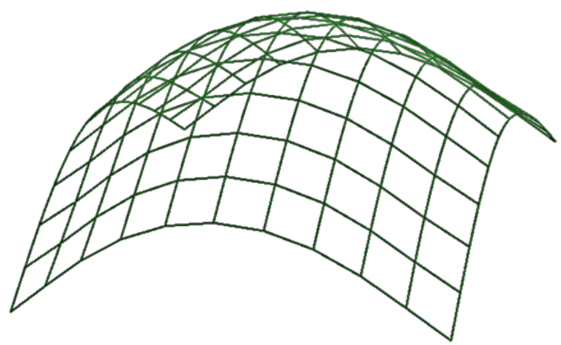
Large deformation analysis

Unknown:

Forced disp., Load,

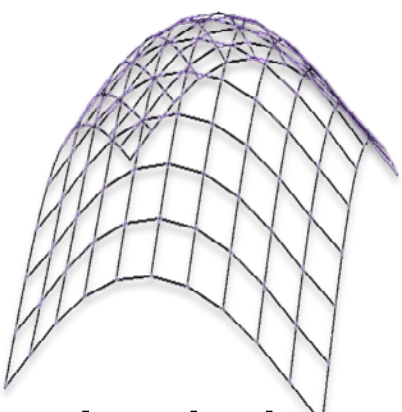
External moment etc…

→ Trial & Error



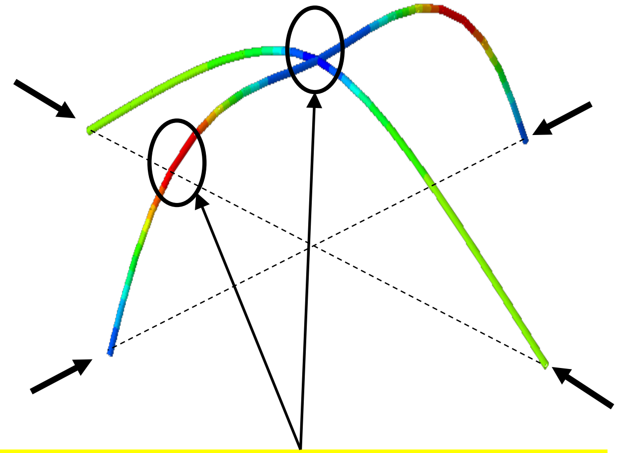
Generated shape

Designer



Desired shape

Inappropriate shape



Interaction forces

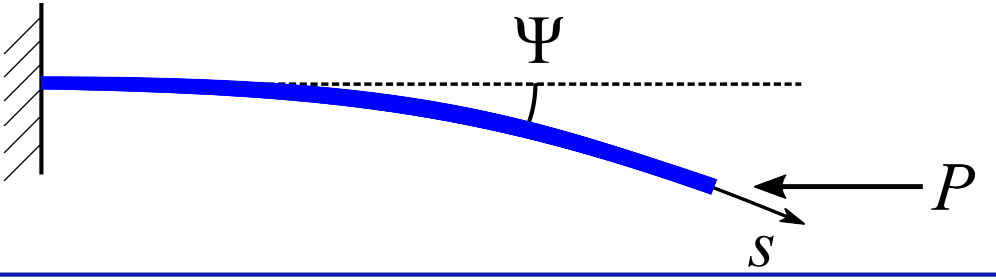
→ Too large

Solution & Background

Target shape of a curved beam

→ Elastica: Shape of a buckled beam-column

→ Reduce the interaction forces [1]



Continuous elastica

$$\text{min.} \quad U = \int \left(\frac{1}{2} EI\kappa^2 + \beta \right) ds \,,$$

$$\kappa(s) = \frac{\partial \Psi(s)}{\partial s}$$

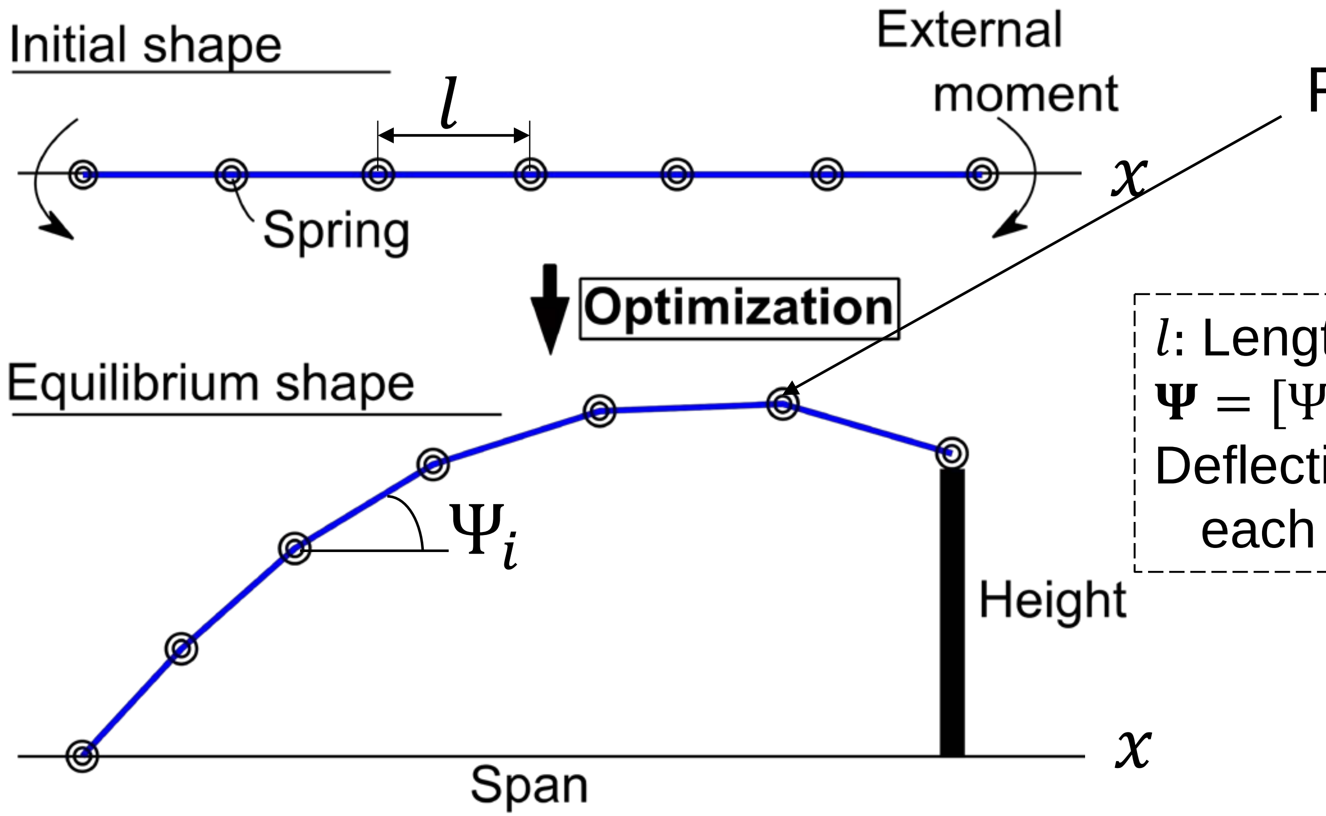
s: arc-length parameter, EI: bending stiffness, $\kappa(s)$: curvature,

β: penalty parameter for the total length

[1] Ohsaki M, Seki K, Miyazu Y. Optimization of locations of slot connections of gridshells modeled using elastica. Proc IASS Symposium 2016, Tokyo. Int Assoc Shell and Spatial Struct 2016;Paper No. CS5A-1012.

Discrete elastica

1. Initial shape (straight) → Equilibrium shape
2. Span & Height of a support → Only a few parameters
3. Uniaxial bending → Planar model



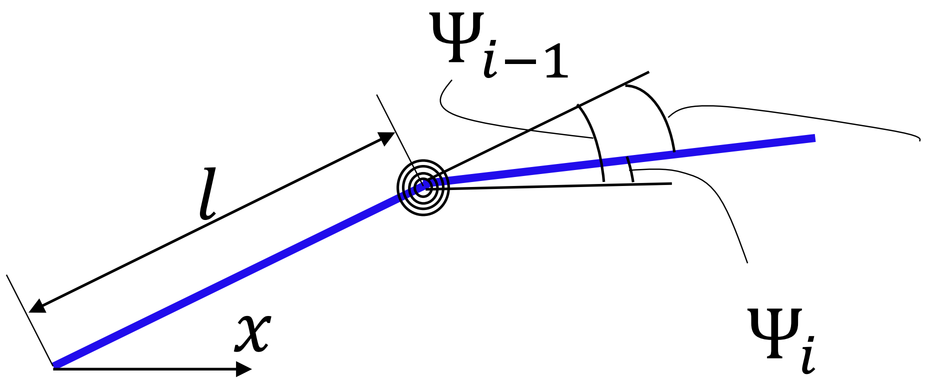
l: Length of each segment,
$\boldsymbol{\Psi} = [\Psi_0, \ldots, \Psi_N]^T$:
Deflection angles between each segment and x axis

[2] A. M. Bruckstein, R. J. Holt and A. N. Netravali, Discrete elastica, Appl. Anal., Vol. 78, pp. 453-485, 2001.

[3] Y. SAKAI and M. OHSAKI, Discrete elastica for shape design of gridshell, Eng. Struct., Vol. 169, pp. 55-67, 2018.

Equivalence of strain energy

→ Stiffness of rotational spring EI/l



$$\theta_i = \Psi_{i-1} - \Psi_i$$

Curvature: $\kappa = \frac{\theta_i}{l}$

Strain energy: $S = \frac{1}{2} EI\kappa^2 l = \frac{1}{2} EI \left(\frac{\theta_i}{l}\right)^2 l = \frac{EI}{2l} \theta_i^2$

P_i $(i = 0, \ldots, N+1)$: Nodes

l: Length of each segment

$\mathbf{\Psi} = (\Psi_0, \ldots, \Psi_N)$: Deflection angle of a segment from x-axis

Optimization problem (Minimize total potential energy)

$$\min_{\boldsymbol{\Psi},\boldsymbol{\Phi},l} \ \Pi(\boldsymbol{\Psi}, l) = \sum_{i=1}^{N} \left[\frac{EI}{2l} \left(\Psi_i - \Psi_{i-1} \right)^2 + \beta l \right]$$

$$- M_0 \Psi_0 - M_{N+1} \Psi_N$$

subject to $\sum_{i=0}^{N} l \cos \Psi_i = L,$: Span length

$\sum_{i=0}^{N} l \sin \Psi_i = H$: Height of a support

(red line) Discretized form of $U = \int \left(\frac{1}{2} EI\kappa^2 + \beta \right) ds$

(blue line) External work

$\beta = 1000$

Stationary conditions = Equilibrium of segments

→ Lagrange multipliers = Reaction forces

Lagrangian (λ_1 and λ_2 are Lagrange multipliers)

$$\mathcal{L}(\boldsymbol{\Psi}, l, \lambda_1, \lambda_2) = \Pi + \lambda_1 \left(\sum_{i=1}^{N} l \cos \Psi_i - L \right) + \lambda_2 \left(\sum_{i=1}^{N} l \sin \Psi_i - H \right)$$

Stationary conditions:

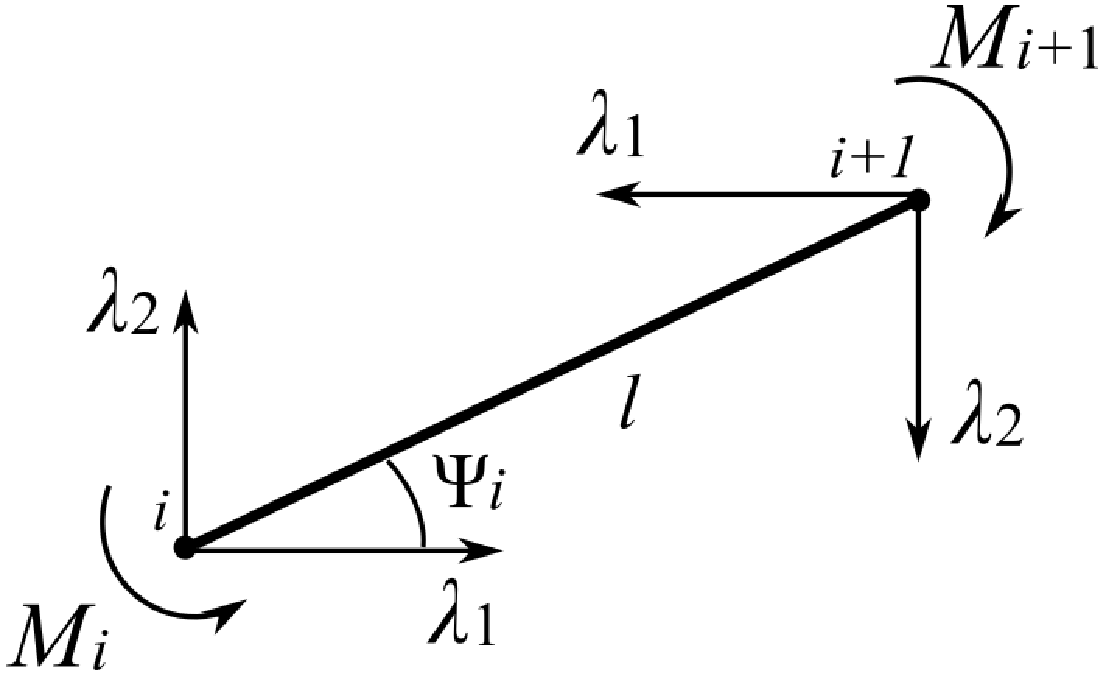
Derivatives with respect to $\boldsymbol{\Psi} = [\Psi_0, \dots, \Psi_N]^T$

$$\frac{EI}{l}\left[-\frac{\Psi_{i-1} - \Psi_i}{l} + \frac{\Psi_i - \Psi_{i+1}}{l}\right] - \lambda_1 \sin \Psi_i + \lambda_2 \cos \Psi_i = 0$$

$(i = 1, \dots, N-1)$

$$\frac{1}{l}\left[\frac{EI(\Psi_1 - \Psi_0)}{l} - M_0\right] - \lambda_1 \sin \Psi_0 + \lambda_2 \cos \Psi_0 = 0$$

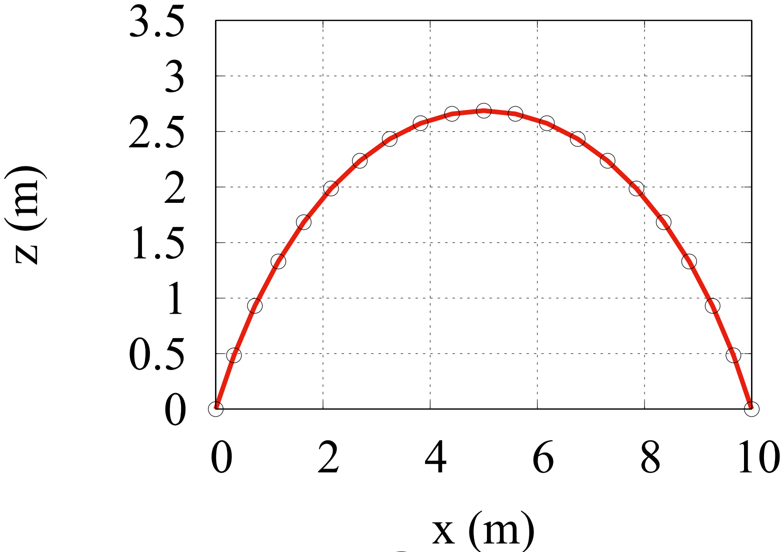
$$\frac{1}{l}\left[\frac{EI(\Psi_N - \Psi_{N-1})}{l} - M_{N+1}\right] - \lambda_1 \sin \Psi_N + \lambda_2 \cos \Psi_N = 0$$



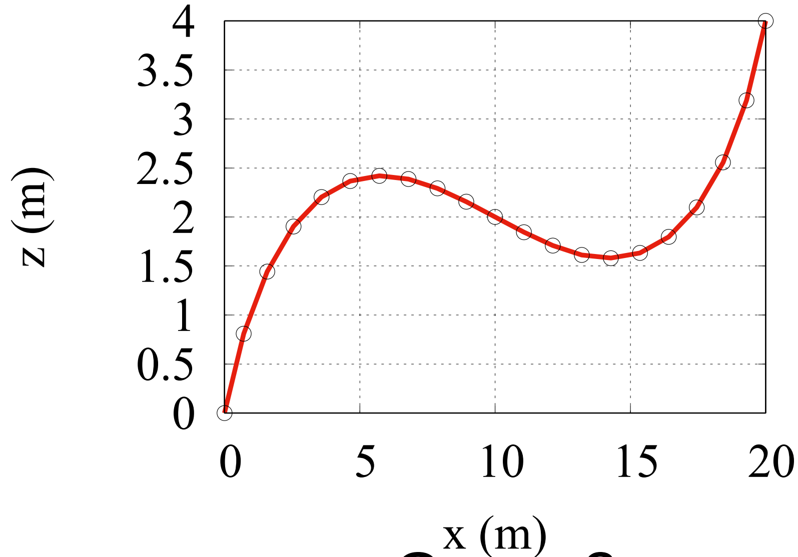
$$\frac{EI}{l}\left[-\frac{\Psi_{i-1}-\Psi_i}{l}+\frac{\Psi_i-\Psi_{i+1}}{l}\right]-\lambda_1\sin\Psi_i+\lambda_2\cos\Psi_i=0$$

→ λ_1 and λ_2: Support reaction forces

Comparisons of shapes and reaction forces



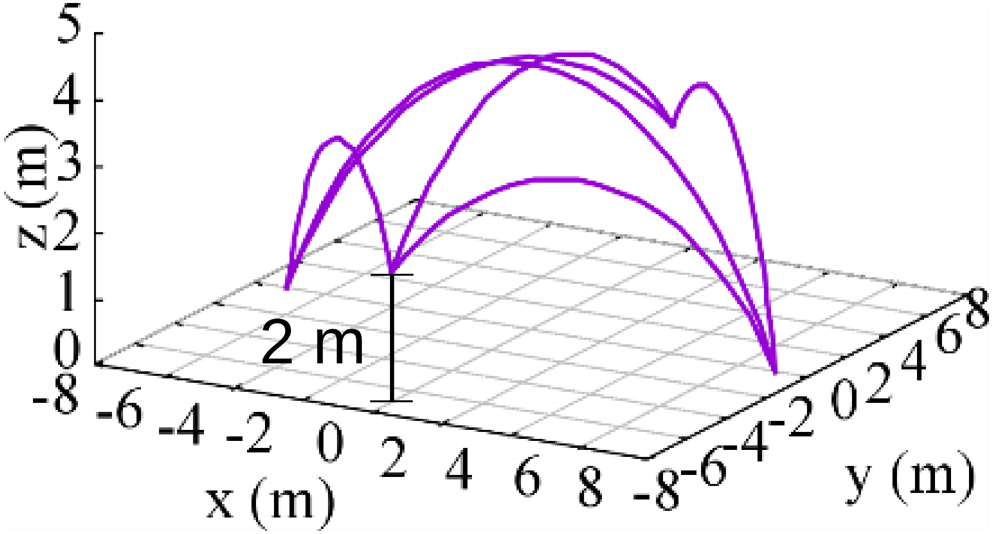
Curve 1



Curve 2

Reaction forces

		Discrete elastica		Continuous beam
Curve 1	λ_1[kN] (x-dir.)	0.4342	$\simeq$	0.4449
	λ_2[kN] (z-dir.)	0.0000	$\simeq$	0.0002
Curve 2	λ_1[kN] (x-dir.)	0.8940	$\simeq$	0.9140
	λ_2[kN] (z-dir.)	0.8212	$\simeq$	0.8171

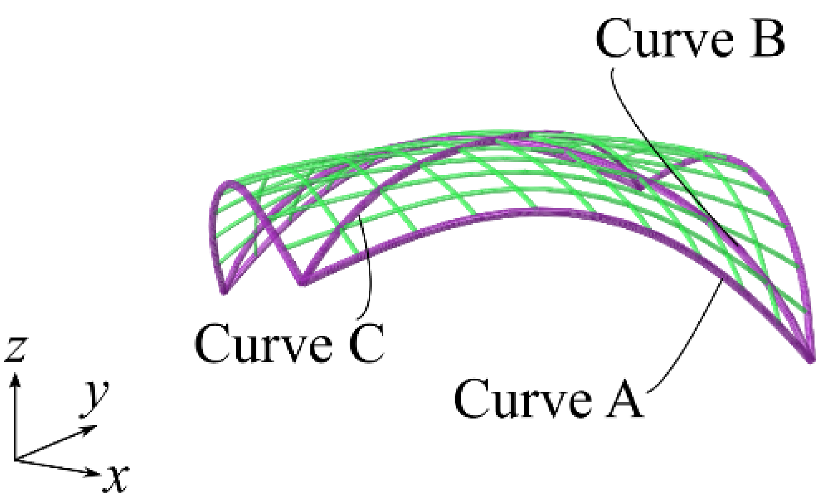


Target surface composed of primary beams
(Discrete elastica)

- Plan:
  Square（10 m×10 m）
- Curve A
  → Boundary

(Height difference = 2m)

- Curve B, C
  → In the diagonal plane



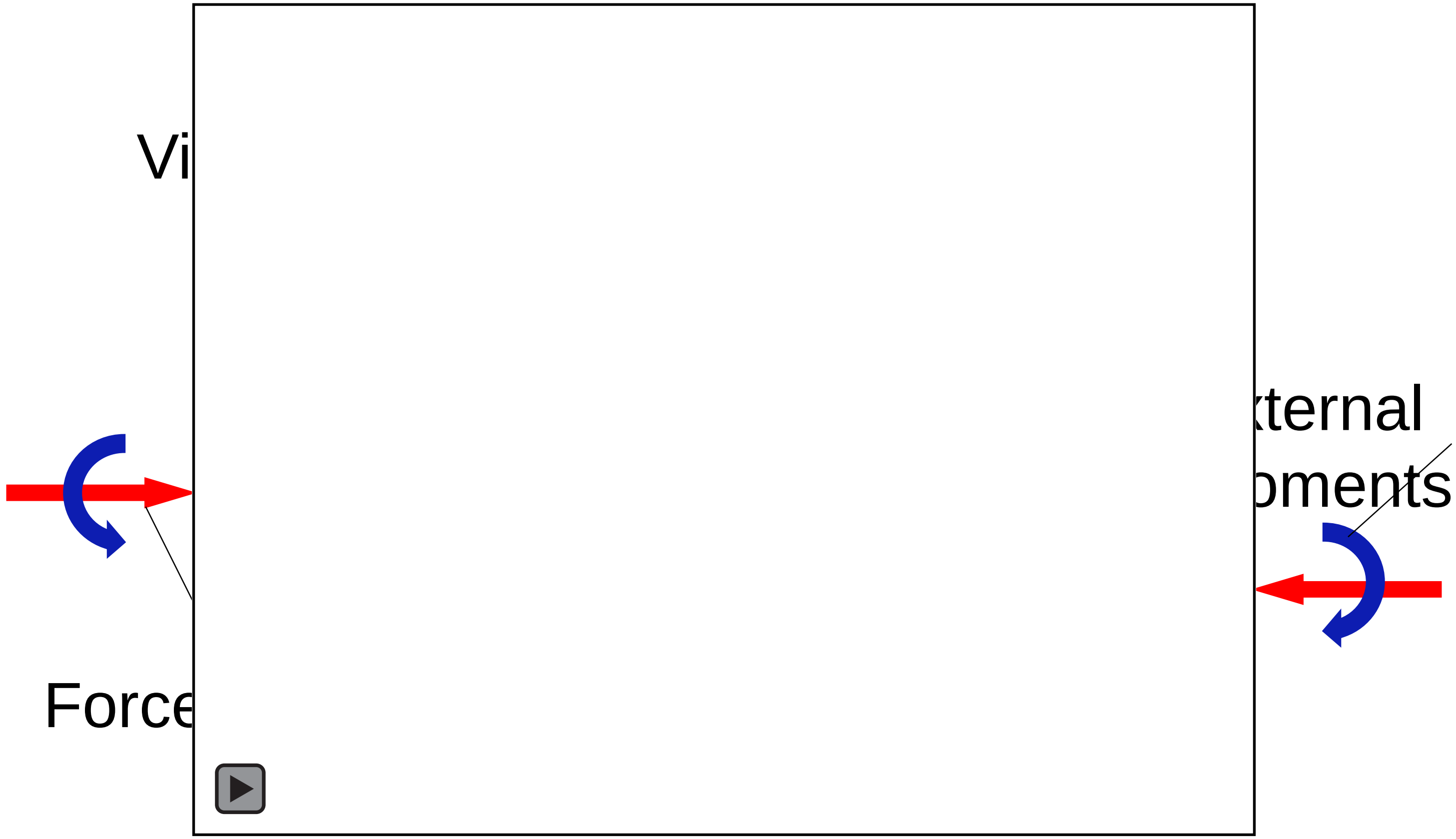
Gridshell (Large deformation analysis)

Material: Glass fiber reinforced polymer （GFRP）

Sectional area: Hollow cyrinder

Yield stress	200 MPa	
	Diameter	Thickness
Primary	0.08 m	0.008 m
Secondary	0.0475 m	0.003m

Motion of large-deformation analysis

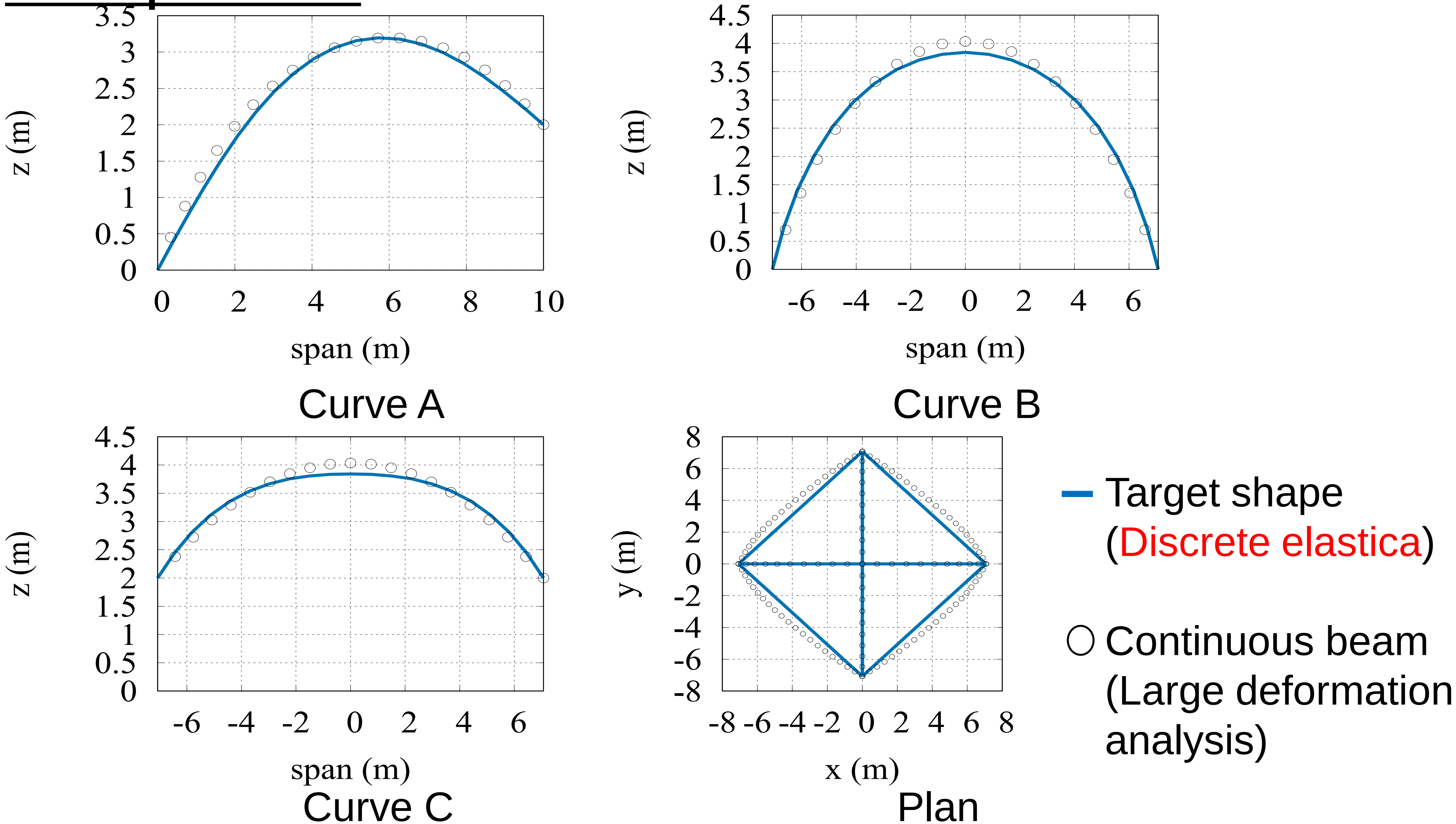


t: Loading parameter $(0.0 \leq t \leq 2.0)$

$0.0 \leq t \leq 1.0$: Upward virtual load (equivalent to self-weight)

$1.0 \leq t \leq 2.0$: Forced displacement & external moments

Comparison



- All member stress < Yielding stress 200 MPa
- Discrete & continuous beams → close

Conclusion

1. Discrete elastica → Equilibrium curved beams derived from optimization → Target shape of primary beams of gridshell
2. Span, height of a support, and external moments → Shape parameters
3. Verification (Primary beams) Target shape vs Large-deformation analysis →Very close

Other studies of gridshell

- Combinatorial optimization for arranging slot+hinge joints
- Spatial discrete elastica (extended from Discrete elastica)
- Development of 3D elastic beam model for large-deformation analysis