Discrete elastica for shape design of gridshells

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Problem of the generating process

Engineer

Initial shape

Large deformation analysis
Unknown: Forced disp., Load, External moment etc…
→ Trial & Error

Designer

Desired shape

Inappropriate shape

Interaction forces → Too large

Generated shape
Solution & Background

Target shape of a curved beam

→ Elastica: Shape of a buckled beam-column

→ Reduce the interaction forces [1]

Continuous elastica

\[
\text{min. } U = \int \left( \frac{1}{2} EI \kappa^2 + \beta \right) ds,
\]

\[
\kappa(s) = \frac{\partial \Psi(s)}{\partial s}
\]

s: arc-length parameter, EI: bending stiffness, \( \kappa(s) \): curvature, \( \beta \): penalty parameter for the total length

1. Initial shape (straight) → Equilibrium shape
2. Span & Height of a support → Only a few parameters
3. Uniaxial bending → Planar model

\[ l: \text{Length of each segment,} \]
\[ \Psi = [\Psi_0, ..., \Psi_N]^T: \]
Deflection angles between each segment and \( x \) axis.

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Equivalence of strain energy
\[ \rightarrow \text{Stiffness of rotational spring} \]

\[ EI/l \]

\[ \begin{align*}
\theta_i &= \Psi_{i-1} - \Psi_i \\
\end{align*} \]

Curvature: \[ \kappa = \frac{\theta_i}{l} \]

Strain energy: \[ S = \frac{1}{2} EI \kappa^2 l = \frac{1}{2} EI \left( \frac{\theta_i}{l} \right)^2 l = \frac{EI}{2l} \theta_i^2 \]

\[ P_i (i = 0, ..., N + 1): \text{Nodes} \]
\[ l: \text{Length of each segment} \]
\[ \Psi = (\Psi_0, ..., \Psi_N): \text{Deflection angle of a segment from } x\text{-axis} \]
## Optimization problem (Minimize total potential energy)

\[
\min_{\Psi, \Phi, l} \Pi(\Psi, l) = \sum_{i=1}^{N} \left[ \frac{EI}{2l} (\Psi_i - \Psi_{i-1})^2 + \beta l \right] - M_0 \Psi_0 - M_{N+1} \Psi_N
\]

subject to \[\sum_{i=0}^{N} l \cos \Psi_i = L, : \text{Span length}\]
\[\sum_{i=0}^{N} l \sin \Psi_i = H : \text{Height of a support}\]

Discretized form of \[U = \int \left( \frac{1}{2} EI \kappa^2 + \beta \right) ds\]

Stationary conditions = Equilibrium of segments  
\[\Rightarrow\] Lagrange multipliers = Reaction forces

\[\beta = 1000\]
Lagrangian ($\lambda_1$ and $\lambda_2$ are Lagrange multipliers)

$$\mathcal{L}(\Psi, l, \lambda_1, \lambda_2) = \Pi + \lambda_1 \left( \sum_{i=1}^{N} l \cos \Psi_i - L \right) + \lambda_2 \left( \sum_{i=1}^{N} l \sin \Psi_i - H \right)$$

Stationary conditions:

Derivatives with respect to $\Psi = [\Psi_0, \ldots, \Psi_N]^T$

$$\frac{EI}{l} \left[ -\frac{\Psi_{i-1} - \Psi_i}{l} + \frac{\Psi_i - \Psi_{i+1}}{l} \right] - \lambda_1 \sin \Psi_i + \lambda_2 \cos \Psi_i = 0$$

$(i = 1, \ldots, N - 1)$

$$\frac{1}{l} \left[ \frac{EI(\Psi_1 - \Psi_0)}{l} - M_0 \right] - \lambda_1 \sin \Psi_0 + \lambda_2 \cos \Psi_0 = 0$$

$$\frac{1}{l} \left[ \frac{EI(\Psi_N - \Psi_{N-1})}{l} - M_{N+1} \right] - \lambda_1 \sin \Psi_N + \lambda_2 \cos \Psi_N = 0$$
\[
\frac{EI}{l} \left[ -\frac{\Psi_{i-1} - \Psi_i}{l} + \frac{\Psi_i - \Psi_{i+1}}{l} \right] - \lambda_1 \sin \Psi_i + \lambda_2 \cos \Psi_i = 0
\]

\[\Rightarrow \lambda_1 \text{ and } \lambda_2: \text{ Support reaction forces} \]
Comparisons of shapes and reaction forces

Curve 1

\[ \lambda_1 \text{[kN]} (x\text{-dir.}) = 0.4342 \]
\[ \lambda_2 \text{[kN]} (z\text{-dir.}) = 0.0000 \]

Curve 2

\[ \lambda_1 \text{[kN]} (x\text{-dir.}) = 0.8940 \]
\[ \lambda_2 \text{[kN]} (z\text{-dir.}) = 0.8212 \]

Reaction forces

<table>
<thead>
<tr>
<th></th>
<th>Discrete elastica</th>
<th>Continuous beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curve 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 ) \text{[kN]} (x\text{-dir.})</td>
<td>0.4342</td>
<td>0.4449</td>
</tr>
<tr>
<td>( \lambda_2 ) \text{[kN]} (z\text{-dir.})</td>
<td>0.0000</td>
<td>0.0002</td>
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<tr>
<td>Curve 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 ) \text{[kN]} (x\text{-dir.})</td>
<td>0.8940</td>
<td>0.9140</td>
</tr>
<tr>
<td>( \lambda_2 ) \text{[kN]} (z\text{-dir.})</td>
<td>0.8212</td>
<td>0.8171</td>
</tr>
</tbody>
</table>
• Plan:
  - Square (10 m × 10 m)

• Curve A
  → Boundary
  (Height difference = 2 m)

• Curve B, C
  → In the diagonal plane

Target surface composed of primary beams
(Discrete elastica)

Material: Glass fiber reinforced polymer (GFRP)

Sectional area: Hollow cylinder

<table>
<thead>
<tr>
<th></th>
<th>Yield stress</th>
<th>Diameter</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary</td>
<td>200 MPa</td>
<td>0.08 m</td>
<td>0.008 m</td>
</tr>
<tr>
<td>Secondary</td>
<td></td>
<td>0.0475 m</td>
<td>0.003 m</td>
</tr>
</tbody>
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Gridshell (Large deformation analysis)
Motion of large-deformation analysis

$t$: Loading parameter $(0.0 \leq t \leq 2.0)$

$0.0 \leq t \leq 1.0$: Upward virtual load (equivalent to self-weight)

$1.0 \leq t \leq 2.0$: Forced displacement & external moments
Comparison

- All member stress < Yielding stress 200 MPa
- Discrete & continuous beams → close
Conclusion

1. **Discrete elastica** → Equilibrium curved beams derived from optimization → Target shape of primary beams of gridshell

2. Span, height of a support, and external moments → Shape parameters

3. Verification (Primary beams)
   Target shape vs Large-deformation analysis → Very close

Other studies of gridshell

- Combinatorial optimization for arranging slot+hinge joints
- Spatial discrete elastica (extended from Discrete elastica)
- Development of 3D elastic beam model for large-deformation analysis