Force density method for simultaneous optimization of geometry and topology of trusses

Makoto Ohsaki and Kazuki Hayashi

Kyoto University

Optimization of trusses

- Cross-section

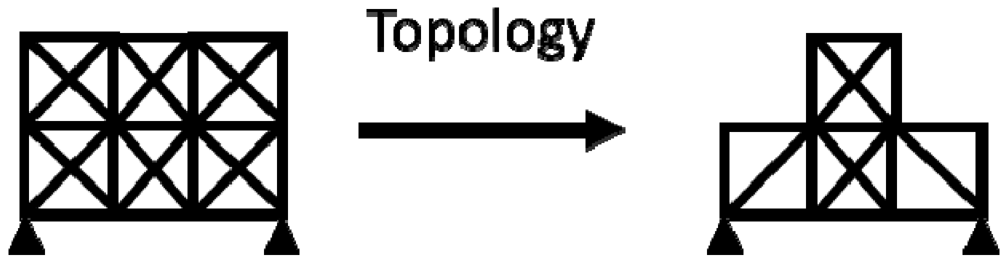
  stiffness

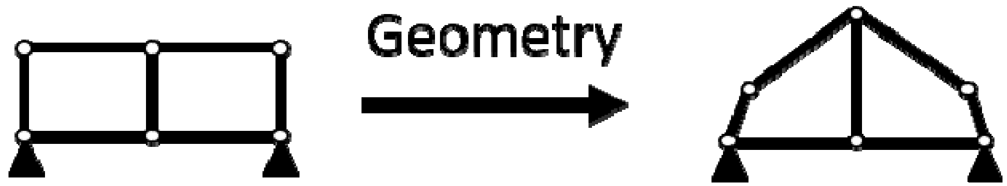
- Topology

  connectivity of nodes and members

- Geometry

  nodal locations





Topology Optimization

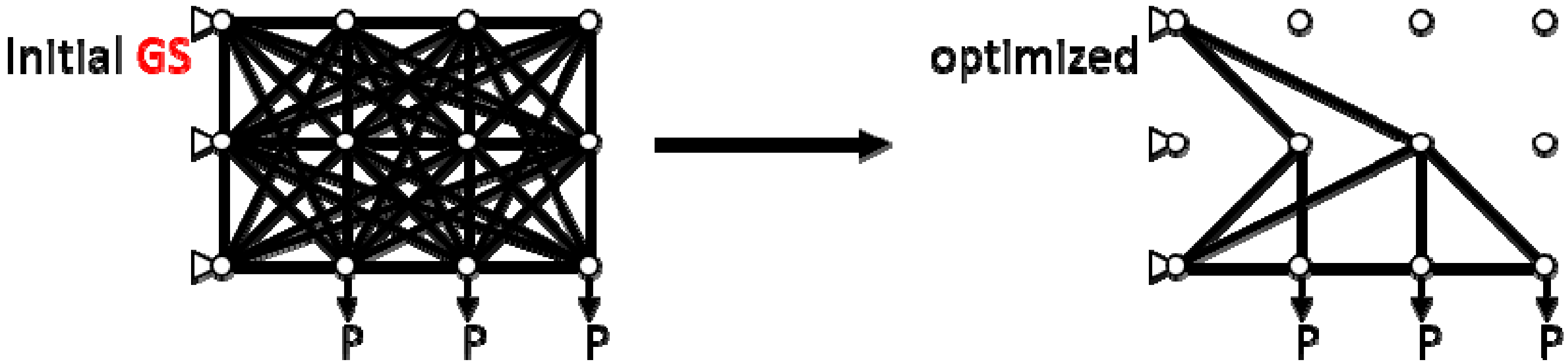
- Ground Structure (GS) Method

  Obtain an optimized sparse truss topology

  from an initial highly connected truss called GS

⟶ Nodal locations are fixed

Dense initial GS to optimize nodal locations



Geometry Optimization

- Design variables: Nodal coordinates
  Add cross-sectional area as design variables

→ Remove thin members to optimize topology

→ Simultaneous optimization of geometry and topology

- Melting nodes exist if wide variation of nodes is allowed

→ Axial stiffness of short member is very large
  Sensitivity coefficients are very large and discontinuous

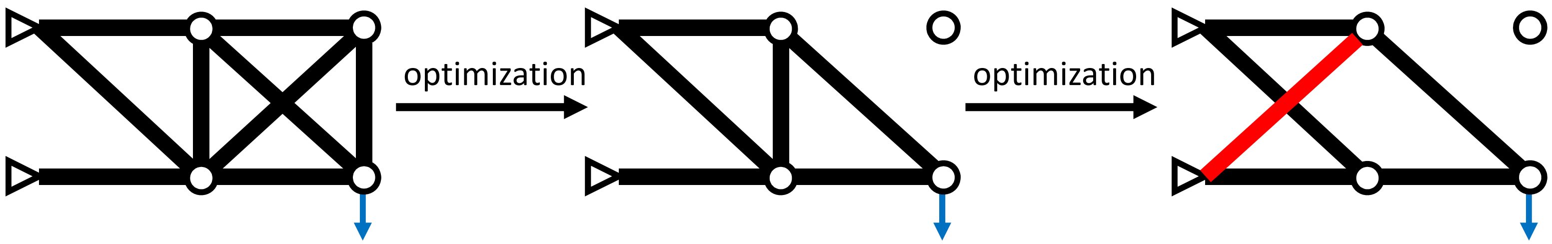
Simultaneous optimization of topology and geometry is very difficult

Existing methods of simultaneous optimization

- Ohsaki (1998):
  Investigate melting nodes for regular grid truss
  Sigmoid function and frame element for smoothing
- Guo, Liu and Li (2003):
  Investigate singularity due to melting nodes for regular truss
- Achtziger (2007):
  Alternating approach of optimization of nodes and cross section
  Constraint to prevent melting nodes
  No melting nodes in numerical examples
- Descamps and Coelho (2014):
  Consider instability constraints using SAND formulation
  Force density is used in intermediate problem but not used in the final formulation

Existing method: Growing method

- Ohsaki and Hagishita (2005)
- Starting from simple GS, adding nodes and members by heuristics
- Optimal solutions with sparse topology and geometry can be obtained
- Cannot satisfy any theoretical optimality criteria



FDM for truss optimization

- Force density:
  FD = (axial force) / (member length)
  Used for equilibrium analysis of tension structures (cable net, tensegrity, membrane structure)
- Easy to avoid problems caused by melting nodes
- Constraint on member length is assigned w.r.t. force density
- Number of variable decrease = number of members (do not include nodal locations)
- Various optimal solutions of geometry and topology can be obtained

Connectivity matrix

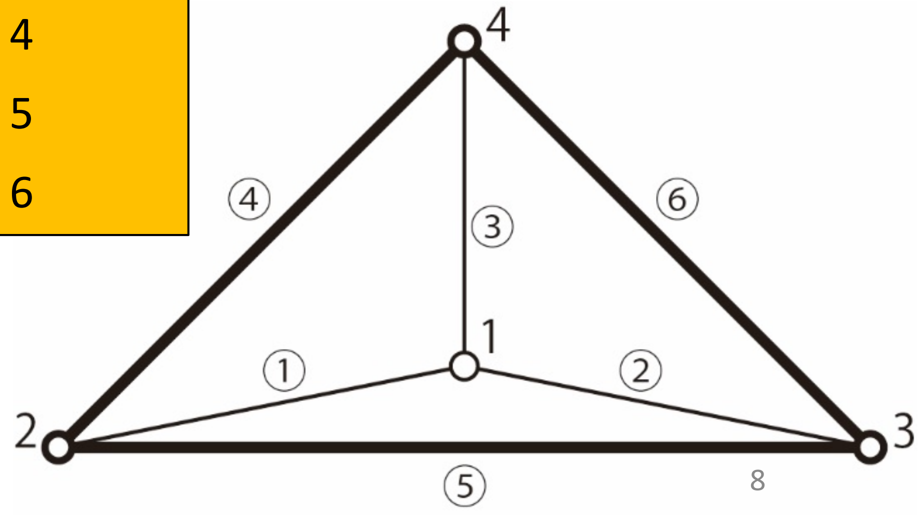
$\boldsymbol{C}$: connectivity matrix・・・express topology of a truss

If member i connects nodes j and k ($j < k$) $\begin{cases} C_{ij} = -1 \\ C_{ik} = 1 \end{cases}$

Node 1 2 3 4

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Member
1
2
3
4
5
6



Force density matrix

q_i : force density = (axial force)/(member length)

$\boldsymbol{Q}$: force density matrix

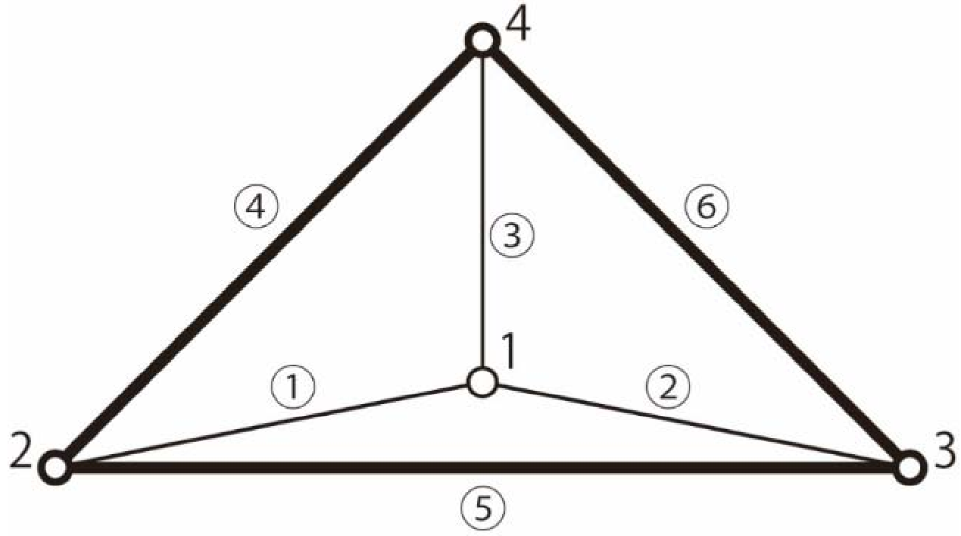
$$\boldsymbol{Q} = \boldsymbol{C}^{\mathrm{T}} diag(\boldsymbol{q})\, \boldsymbol{C}$$

$\boldsymbol{QX} = \boldsymbol{P}$ **:** equilibrium equation w,r,t, nodal coordinates

$\boldsymbol{X}$: nodal coordinates

$\boldsymbol{P}$: nodal load vectors including reactions

Force density matrix



$$\boldsymbol{Q} = \begin{array}{c} \\ \end{array}$$

node 1   node 2   node 3   node 4

$$\begin{bmatrix} q_1 + q_2 + q_3 & -q_1 & -q_2 & -q_3 \\ & q_1 + q_4 + q_5 & -q_5 & -q_4 \\ & & q_2 + q_5 + q_6 & -q_6 \\ & Sym. & & q_3 + q_4 + q_6 \end{bmatrix} \begin{array}{l} \text{node 1} \\ \text{node 2} \\ \text{node 3} \\ \text{node 4} \end{array}$$

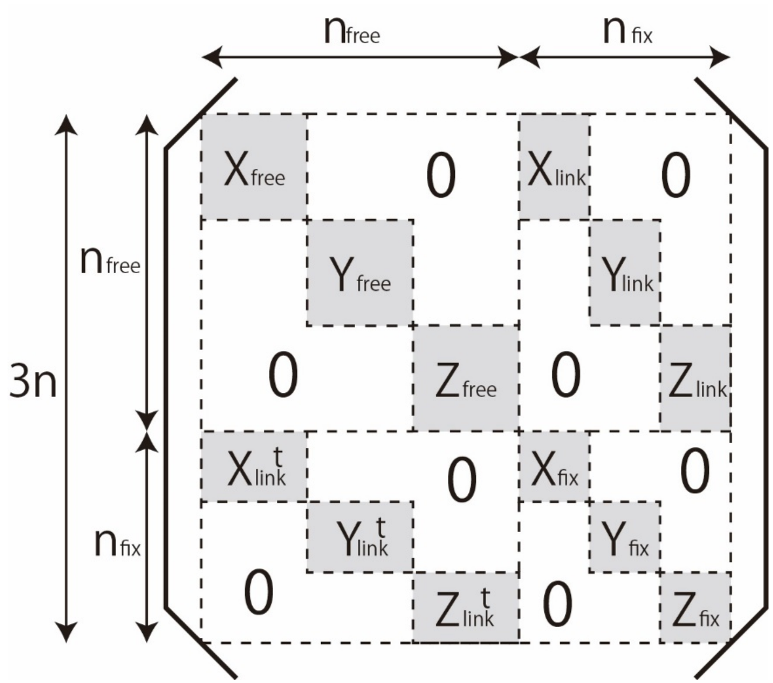
Assemblage of Q in 3-directions

$\tilde{Q}$: force density matrix decomposed into fixed and free elements

n_{free} : No. of free coordinates

n_{fix} : No. of fixed coordinates

$3n$: number of all coordinates (3 × number of nodes)



Loaded nodes are included in fixed nodes

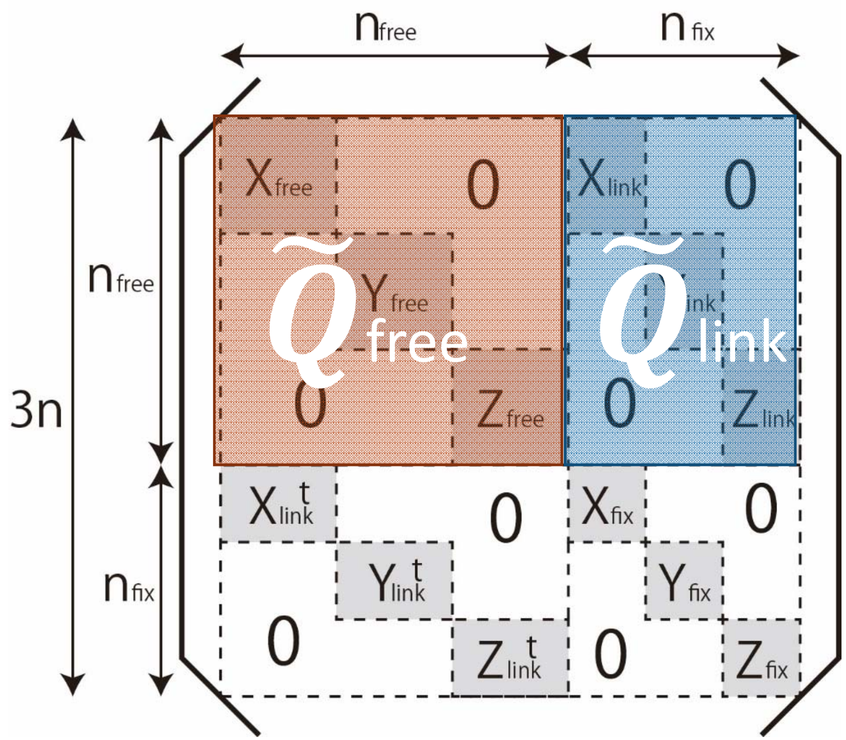
Nodal coordinates

$\tilde{\boldsymbol{Q}}_{\text{free}}$: $\tilde{\boldsymbol{Q}}$ for free components

$\tilde{\boldsymbol{Q}}_{\text{link}}$: $\tilde{\boldsymbol{Q}}$ for link components

$\boldsymbol{X}_{\text{free}}$: free nodal coordinates

$\boldsymbol{X}_{\text{fix}}$: fixed nodal coordinates



Equilibrium equation

$$\tilde{\boldsymbol{Q}}_{\text{free}} \boldsymbol{X}_{\text{free}} = -\tilde{\boldsymbol{Q}}_{\text{link}} \boldsymbol{X}_{\text{fix}}$$

Solve for $\boldsymbol{X}_{\text{free}}$

→ $\boldsymbol{X}_{\text{free}}$ is a function of force density

Equilibrium equation is almost always regular

Structural volume

Minimize compliance under volume constraint

→ Optimal solution is statically determinate

All existing members have same absolute value of stress

$\bar{\sigma}$: absolute value of stress

L_i : length of ith member

V_i : volume of ith member

$$V_i = \frac{|q_i| L_i^2}{\bar{\sigma}}$$

Compliance

E : Young's modulus

A_i : cross-sectional area of ith member

S_i : strain energy of ith member

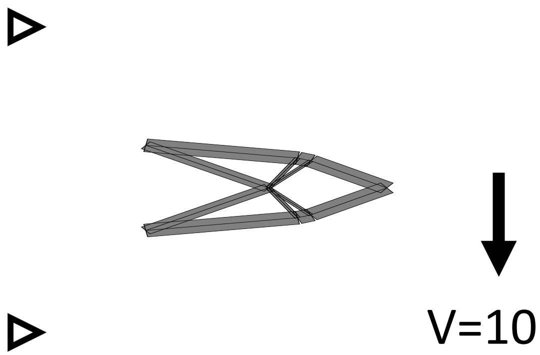
$$A_i = \frac{|N_i|}{\bar{\sigma}} = \frac{|q_i|L_i}{\bar{\sigma}} \quad , \quad S_i = \frac{A_i L_i \bar{\sigma}^2}{2E} = \frac{\bar{\sigma}|q_i|L_i^2}{2E}$$

F : Compliance (objective function)

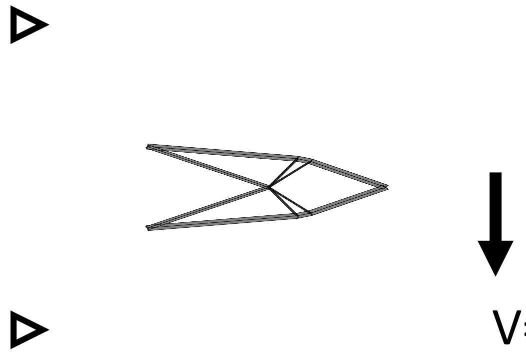
$$F = 2S_i = \sum_{i=1}^{m} \frac{\bar{\sigma}|q_i|L_i^2}{E}$$

Constraint function(V) × Objective function(F)

$$VF = \sum_{i=1}^{m} \frac{q_i^2 L_i^4}{E} = \sum_{i=1}^{m} \frac{N_i^2 L_i^2}{E} = const.$$



V=10
F=1



V=1
F=10

⇒ total volume can be calculated after minimizing compliance for $\bar{\sigma}$

Optimization problem

q_i^U , q_i^L : lower and upper bounds of force density

R_i : reaction force including nodal loads

$\mathcal{R}$: set of nodal load reaction components to be specified

$$\min_{\boldsymbol{q}} F(\boldsymbol{q}) = \sum_{i=1}^{m} \frac{\bar{\sigma}|q_i|L_i^2}{E}$$

$$\text{subject to } \; q_i^L \leq q_i \leq q_i^U \;\; (i = 1, \dots, m)$$

$$R_i(\boldsymbol{q}) = \bar{R}_i \;\; (i \in \mathcal{R})$$

No. of variables = No. of members

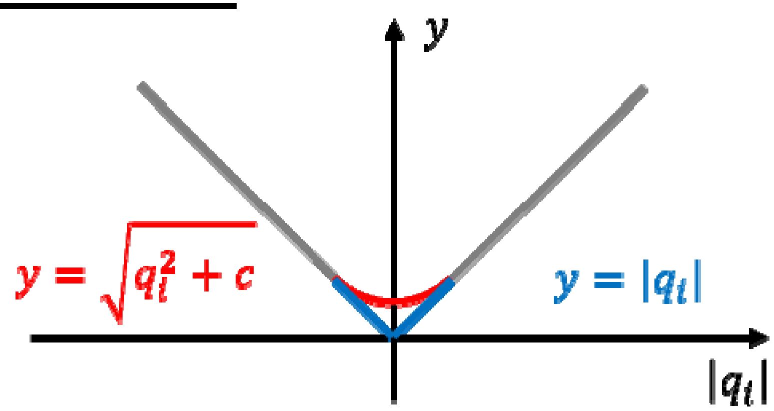
No. of constraints = No. of load components

Smoothing approximation

$|q_i|$: sensitivity is discontinuous at $q_i = 0$

$\rightarrow |q_i| = \sqrt{q_i^2 + c}$

($c > 0$, small positive)



$$\min_{\boldsymbol{q}} F(\boldsymbol{q}) = \sum_{i=1}^{m} \frac{\bar{\sigma} L_i^2 \sqrt{q_i^2 + c}}{E}$$

subject to $q_i^L \leq q_i \leq q_i^U \ \ (i = 1, \dots, m)$

$$R_i(\boldsymbol{q}) = \bar{R}_i \ \ (i \in \mathcal{R})$$

Refinement of optimal solution

Solution may include overlapped nodes and unnecessary members

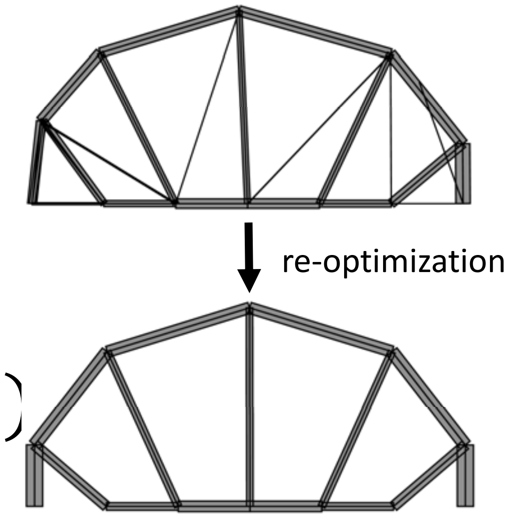
$\longrightarrow$ Geometry and cross-sectional areas are refined

$$\min_{\boldsymbol{X},\boldsymbol{A}} F(\boldsymbol{X}, \boldsymbol{A}) = \sum_{i=1}^{m} \frac{N_i^2 L_i}{E A_i}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} A_i L_i \leq \bar{V}$$

$$X_j^L \leq X_j \leq X_j^U \quad (j = 1, \ldots, n_{free})$$

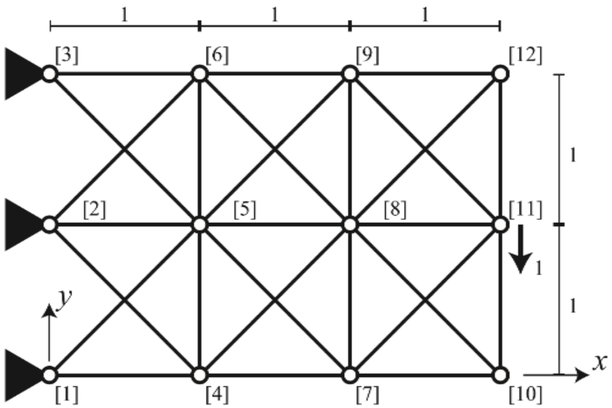
$$A_i^L \leq A_i \leq A_i^U \quad (i = 1, \ldots, m)$$



Numerical examples

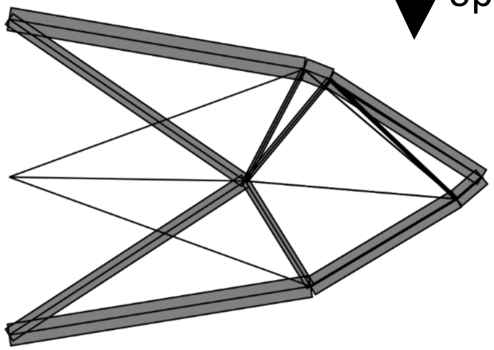
- $E=1.0$, $\bar{\sigma} = 1.0$, $c=1.0 \times 10^{-6}$
- Randomly generate 100 sets of initial values of force density
- Range: $[\bar{q}_i - \delta q,\ \bar{q}_i + \delta q]$
  ($\bar{q}_i$: force density of regular truss with uniform cross-sectional areas)
- SNOPT Ver.7.2 is used for solving NLP problems

3 x 2 truss



	Max	median	min	Ave.	std. dev.
F	10.227	9.095	8.316	9.218	0.549

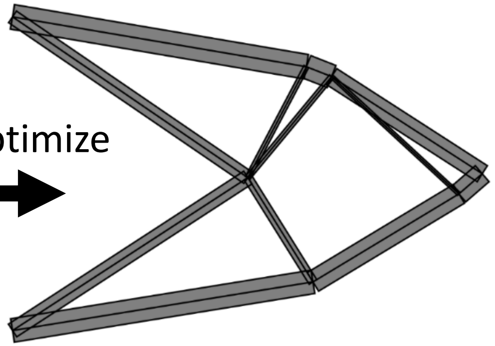
optimize



$F = 8.316$

n = 8

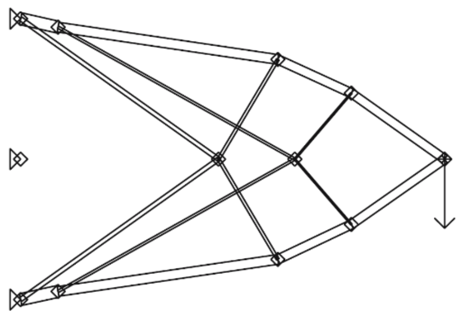
re-optimize



$F = 8.312$

n = 7

Achtziger(2007)

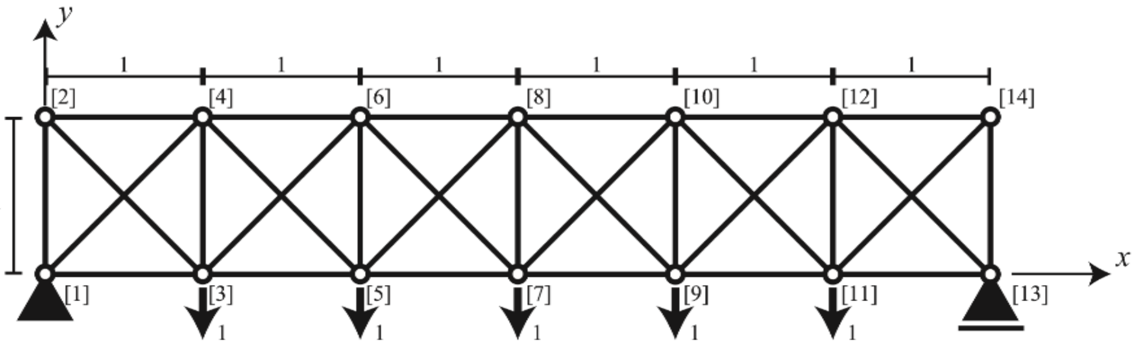


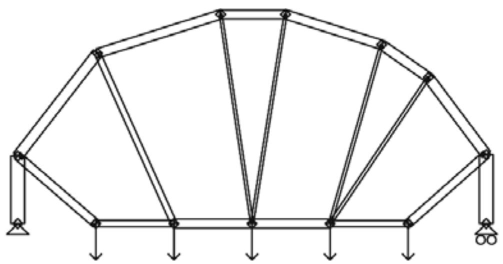
$F = 8.307$

n = 12

6 x 1 truss

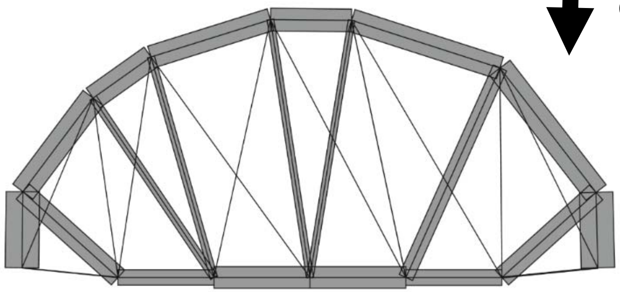
	max	median	min	average	std. dev.
F	7.850×10^6	640.150	118.994	9.271×10^6	7.875×10^6





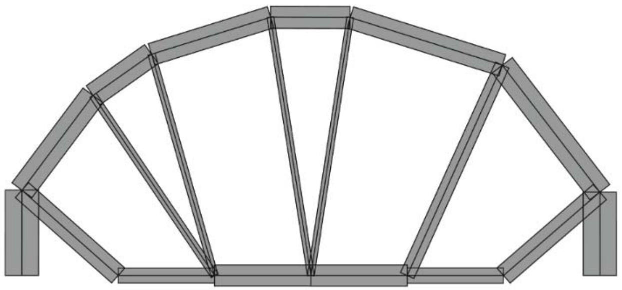
Achtziger(2009) $F = 122.447$

optimize



$F = 118.994$

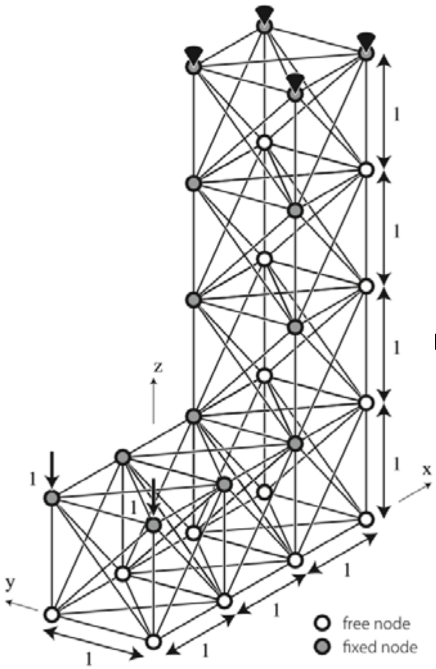
refine



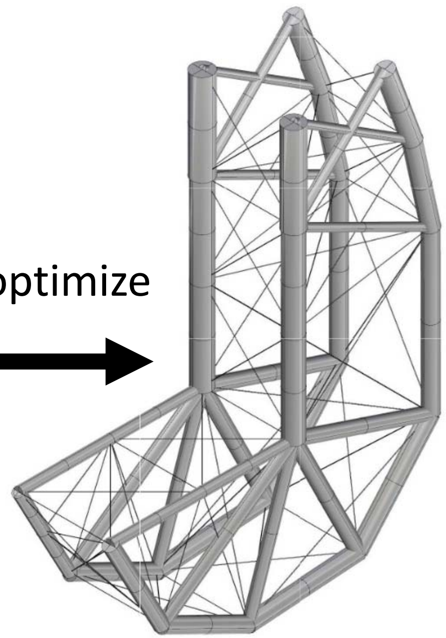
$F = 122.411$

L-shaped truss

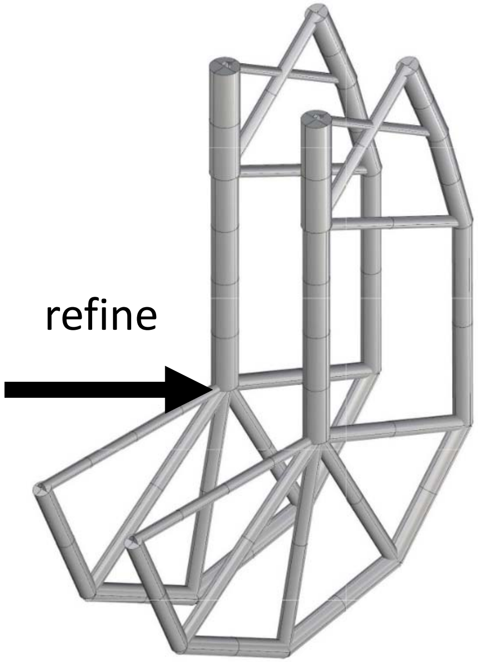
	max	median	min	average	std. dev.
V	4015.431	54.927	54.544	125.171	435.744

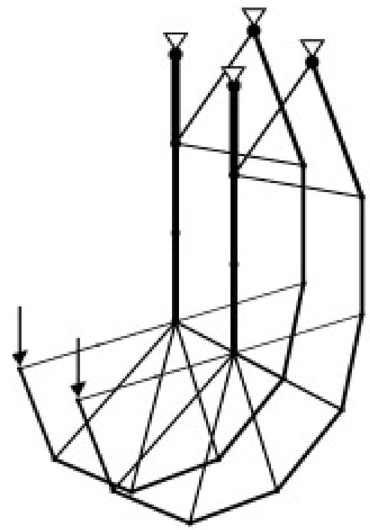


optimize



refine





$V = 54.544$

$V = 54.276$

$V = 54.098$
(Descamps and Coelho (2014))

Conclusions

1. Difficulty in simultaneous optimization of geometry and topology can be successfully avoided using force density as design variable.
2. Compliance and structural volume are expressed as functions of force density only.
   $\longrightarrow$ number of design variables is equal to number of members
3. Discontinuity of the objective function and the sensitivity coefficients w.r.t. force density can be successfully avoided using smoothing function.

Ohsaki and Hayashi, SMO Journal, published online