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A Random Sampling Approach to Worst-case Design of Structures

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Abstract A random sampling approach is presented for worst-case design of structures. Uncertainties are considered in structural and material parameters, which are assumed to exist in intervals with prescribed upper and lower bounds. Constraints are given for the worst responses that are found by solving anti-optimization problems. Optimal cross-sections are then selected from the list of available sections. The regions of uncertainty of parameters are discretized into integer values to formulate the hybrid problem of optimization and anti-optimization as an integer programming problem. The accuracy of solution is defined based on the order of the objective value; hence, a random sampling approach is successfully applied to obtain optimal and anti-optimal solutions within the prescribed accuracy. It is shown in the numerical examples that a good approximate optimal solution is found by random sampling with small number of analyses.

Keywords Optimization · Random sampling · Anti-optimization · Worst-case design · Building frame

1 Introduction

In the conventional structural optimization methods, the parameters representing the structural and material properties are given deterministically. However, in the practical design process, uncertainty in those parameters should be appropriately taken into account (Elishakoff and Ohsaki, 2010). There are various approaches to such purpose; namely, reliability-based approach (Frangopol, 1995; Valdebenito and Scuëller, 2010; Noh *et al.*, 2009), probabilistic approach (Augusti *et al.*, 1984), worst-case design (Rustem and Howe, 2002), and robust design (Gu *et al.*, 2000; Kanno and Guo, 2010). In this paper, we utilize the concept of unknown-but-bounded (Elishakoff *et al.*, 1994), and assume that the uncertain parameters exist in the specified bounded intervals. Constraints are assigned on the worst values of the structural responses. In this case, the optimization problem turns out to be a two-stage problem of optimization and anti-optimization. The optimal design variables are obtained by solving the upper-level optimization problem. The worst parameter values in bounded intervals of the lower-level anti-optimization problem can be found using the standard approach of interval analysis (Moore, 1966); however, this approach is not applicable to a problem with large number of parameters and/or highly nonlinear constraint functions.

Heuristic approaches have been developed for obtaining approximate optimal solutions within reasonable computational cost. They are also called statistical approaches, evolutionary methods, etc., and usually involve randomness. They can be classified into population based approaches and those based on local search (Ohsaki, 2010). The former include genetic algorithms (GA) (Goldberg, 1989; Ohsaki, 1995), and particle swarm optimization (Kennedy, 1997; Schutte and Groenwold, 2003), which have a family of solutions called generation at each step of optimization; therefore, the functions should be evaluated many times before reaching an approximate optimal solution. Furthermore, evaluation of objective and constraint functions requires much computational cost for structural

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optimization problems; therefore, we cannot carry out function evaluation many times for optimization.

Among various approaches based on local search, tabu search (TS) (Glover, 1989) is regarded as a deterministic approach, because it moves to the best solution among all the neighborhood solutions of the current solution, although a randomness exists in the selection of initial solution. A tabu list is used to prevent a cyclic selection of a set of small number of solutions. However, for a problem with many variables, it is not desirable to carry out exhaustive local search at each step. Therefore, we can limit the number of neighborhood solutions in a similar manner as simulated annealing (Aarts and Korst, 1989) and random search (Ohsaki, 2001), which is extensively used in chemical engineering (Luus and Jaakola, 1973; Salcedo *et al.*, 1990).

Recently, random sampling (RS) approach, or randomized algorithm (Mitzenmacher and Upfal, 2005; Lipton and Naughton, 1995), has been studied extensively for knowledge discovery (Domingo *et al.*, 1999), estimation of average and worst computational costs of an algorithm, and finding an approximate optimal solution of a combinatorial problem. Based on the simple formulas of probability theory, we can estimate the probability of obtaining the optimal solution using RS. However, application of RS to a two-stage optimization problem involving real variables has not been investigated.

In this paper, we present a new approach to two-stage problem of optimization and anti-optimization of structures, which is also called worst-case design problem. Uncertainties are considered in structural and material parameters. The parameters are assumed to exist in bounded intervals. A lower-level anti-optimization problem is formulated for finding the worst value of the response. Optimal cross-sections are then selected in the upper-level problem from the list of available sections under constraints on worst responses. The intervals of uncertain parameters are discretized into integer values. Thus, the anti-optimization problem as well as the optimization problem turns out to be an integer programming problem, which is also called a combinatorial problem.

The accuracy of solution is defined based on the order of the objective value; hence, an RS approach is successfully applied to obtain optimal and anti-optimal solutions within the prescribed accuracy. Since we do not carry out statistical investigation, our purpose is different from that of *order statistics* (Hosking, 1990). A mathematical problem is first solved for verification of the proposed approach. Then, optimal cross-sections of a building frame are found under constraint on the worst representative elastoplastic response under seismic motions. It is shown in the numerical examples that a good approximate optimal solution can be successfully found by RS within practically acceptable number of analyses. The results are compared with those by GA and TS.

2 Optimization Problem

Consider a problem of optimizing the cross-sections of structures, which are selected from the pre-assigned list of standard sections. The members are classified into m groups, each of which has the same section. The design variable vector is denoted by $\mathbf{J} = (J_1, \dots, J_m)$, which has integer values. For example, if $J_i = k$, then the members in the ith group have the kth section of the list. Let $F(\mathbf{J})$ denote the objective function representing, e.g., the total structural volume. The constraint functions defined by structural responses are denoted by $G_i(\mathbf{J})$ $(i = 1, \dots, n)$, where n is the number of constraints. Then, the optimization problem is formulated as

$$\begin{aligned} &\text{Minimize} \quad F(\mathbf{J}) && \text{(1a)}\\ &\text{subject to} \quad G_i(\mathbf{J}) \le \bar{G}_i, \quad (i = 1, \dots, n) && \text{(1b)}\\ &\qquad\qquad J_i \in \{1, \dots, s\}, \quad (i = 1, \dots, m) && \text{(1c)} \end{aligned}$$

where $\bar{G}_i$ is the upper bound for G_i, and s is the number of sampling values of variables.

We incorporate uncertainty in structural and material parameters such as the geometry of member cross-section, location of node, Young's modulus, and yield stress. The vector consisting of uncertain parameters is denoted by $\mathbf{p} = (p_1, \dots, p_r)$, where r is the number of parameters. Then, the structural response is given as a function of $\mathbf{J}$ and $\mathbf{p}$ as $\tilde{G}_i(\mathbf{J}, \mathbf{p})$. We assign constraints on the worst values $\hat{G}_i(\mathbf{J})$ of responses, and formulate the optimization problem as

$$\begin{aligned} &\text{Minimize} \quad F(\mathbf{J}) && \text{(2a)}\\ &\text{subject to} \quad \hat{G}_i(\mathbf{J}) \le \bar{G}_i, \quad (i = 1, \dots, n) && \text{(2b)}\\ &\qquad\qquad J_i \in \{1, \dots, s\}, \quad (i = 1, \dots, m) && \text{(2c)} \end{aligned}$$

The worst value $\hat{G}_i(\mathbf{J})$ is obtained by solving the following anti-optimization problem:

$$\begin{aligned} &\text{Find} \quad \hat{G}_i(\mathbf{J}) = \max_{\mathbf{p}} \tilde{G}_i(\mathbf{J}, \mathbf{p}) && \text{(3a)}\\ &\text{subject to} \quad \mathbf{p}^{\mathrm{L}} \le \mathbf{p} \le \mathbf{p}^{\mathrm{U}} && \text{(3b)} \end{aligned}$$

where $\mathbf{p}^{\mathrm{U}} = (p_1^{\mathrm{U}}, \dots, p_r^{\mathrm{U}})$ and $\mathbf{p}^{\mathrm{L}} = (p_1^{\mathrm{L}}, \dots, p_r^{\mathrm{L}})$ are the upper and lower bounds for $\mathbf{p}$, respectively, which can be obtained from measurements, statistics, and experiments. Hence, the optimal solution considering the worst values of responses can be found by solving a two-stage problem of optimization and anti-optimization.

3 Optimization methods

3.1 Tabu search

The simplest heuristic approach may be the local random search that consecutively selects the best solution in the neighborhood of the current solution. The convergence property to the local optimal solution may be enhanced if many solutions, or preferably all neighborhood solutions, are searched to select the best solution. A neighborhood solution that does not improve the objective value can also be selected to reduce the possibility of being trapped at a local optimal solution. However, in this case, a so-called cycling or loop can occur, where a set of neighboring solutions is chosen iteratively. TS has been developed to prevent cycling utilizing the tabu list containing the prohibited solutions that have already been searched (Glover, 1989).

Suppose the constraints are incorporated into the objective function using a penalty function approach. The algorithm of TS for a minimization problem is shown in Fig. 1, where N^{b} is the number of neighborhood solutions, N^{s} is the number of steps, N^{t} is the maximum size of tabu list, and the superscript (k) denotes a value at the kth iteration.

In the following examples, we apply TS to both optimization and anti-optimization problems for comparison purpose with the random sampling approach presented below. The neighborhood solutions are generated using a random number $\tau \in [0, 1]$. Each variable is increased if $\tau \geq 2/3$, decreased if $\tau < 1/3$, and is not modified if $1/3 \leq \tau < 2/3$.

3.2 Random sampling

We first discuss applicability of RS to an anti-optimization problem. Consider a maximization problem with r uncertain parameters, which can take q different integer values. The total number of different parameter sets is $M = r^q$. When solving an anti-optimization problem, it often happens that the global worst solution is not necessary to be found, and only a good approximate solution is needed because the bounds of parameters are not defined rigorously.

The accuracy of the worst solution may be defined using the objective function value; however, we cannot use this approach because the global worst value is unknown. Therefore, we define the accuracy of solution based on the order of objective value. Although we do not carry out enumeration when using the RS approach, all solutions are supposed to be enumerated and numbered in non-increasing order of the objective function value, for discussing the accuracy of the solution; i.e.,

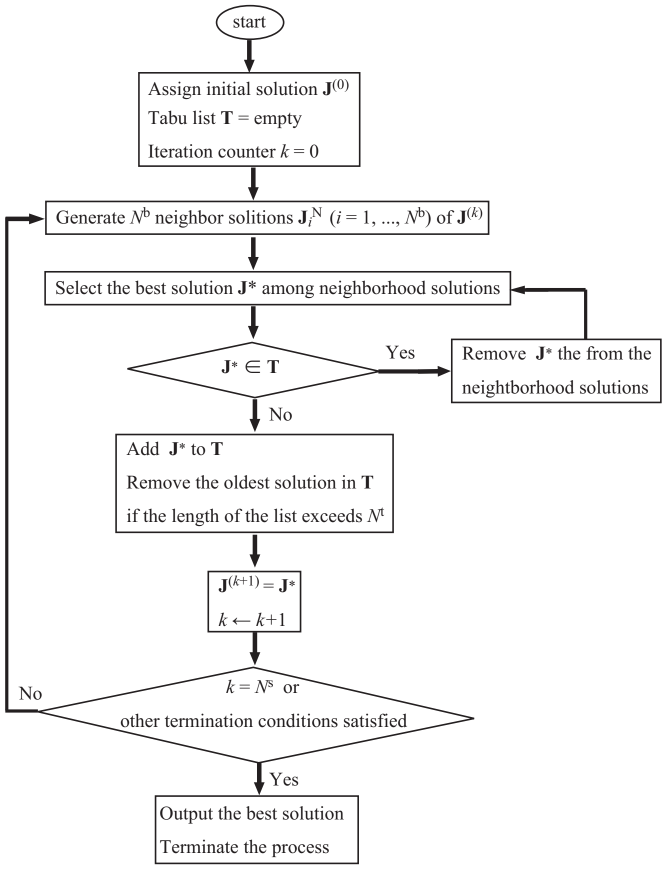

Fig. 1 Algorithm of TS for a minimization problem.

the first solution has the worst (maximum) objective value.

We assume that N worst solutions are regarded as approximate worst solutions. Suppose the parameters p_i $(i = 1, \ldots, r)$ have the uniform probability density functions

$$\phi(p_i) = \frac{1}{p_i^{\mathrm{U}} - p_i^{\mathrm{L}}}, \quad (i = 1, \ldots, r) \tag{4}$$

The parameter p_i is sampled to q values p_{ij} $(j = 1, \ldots, q)$ using an integer value I_i that can take q values I_{ij} $(j = 1, \ldots, q)$ as

$$p_{ij} = (I_{ij} - 0.5)\frac{p_i^{\mathrm{U}} - p_i^{\mathrm{L}}}{q}, \quad (i = 1, \ldots, r; j = 1, \ldots, q) \tag{5}$$

Then, I_{ij} has the same probability $1/q$ to be sampled. Therefore, the probability of failing to obtain an approximate worst solution is $1 - N/M$ for a randomly sampled solution with uniform probability from the set of M solutions. Hence, the probability that no approximate worst solution is found after t random selections is $(1 - N/M)^t$. Note that we utilize the procedure of

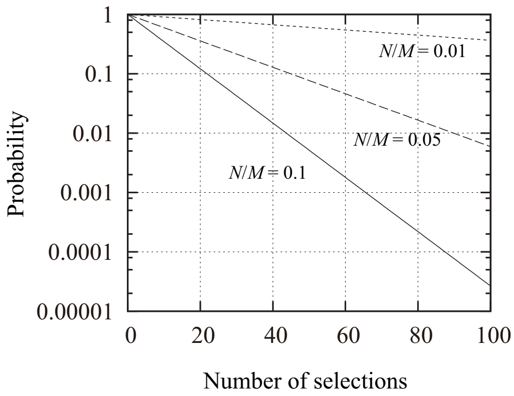

Fig. 2 Probability of failing to obtain an approximate worst solution.

sampling with replacement; i.e., a solution might be selected more than once, and the probability of selecting a particular solution is always $1/M$.

Fig. 2 shows the probabilities of failing to obtain approximate worst solutions for $N/M = 0.1$, 0.05, and 0.01. Note that the vertical axis has a logarithmic scale. For example, the probability for $N/M = 0.05$ is less than 0.01 if random sampling is carried out 100 times. It should be noted here that the relation does not depend on M directly. The required number of analysis for the desired probability depends on the ratio N/M.

We can modify the values of I_{ij}, if the probability functions of parameters are known to be nonuniform. Using the well known fact that the probability density of a probability distribution function $\Phi(p_i)$ is uniform in the interval [0,1], I_{ij} can be given as follows:

$$I_{ij} = \Phi^{-1}(k_{ij}), \quad k_{ij} = (j - 0.5)\frac{1}{q} \tag{6}$$

However, the probability distribution is not important if only an approximate worst solution is to be found and the variations of responses are not investigated.

For the upper-level optimization problem, it is natural to define the accuracy of solution using the order of solutions, because it is an combinatorial problem. This way, the order of solutions can be utilized for both optimization and anti-optimization problems, and the two-stage problem is solved using an RS approach. Let A^{o} and A^{a} denote the number of random sets to be generated, respectively, in optimization and anti-optimization problems, and let $\mathbf{I} = (I_1, \ldots, I_r)$. The RS algorithm is summarized in Fig. 3.

Although the purpose of this paper is to present a computationally inexpensive method for approximate worst-case design, we can carry out probabilistic evaluation of the objective function, if a large number of samples are available. Let $\tilde{f}$ denote the ratio of approximate worst solutions in the M original samples.

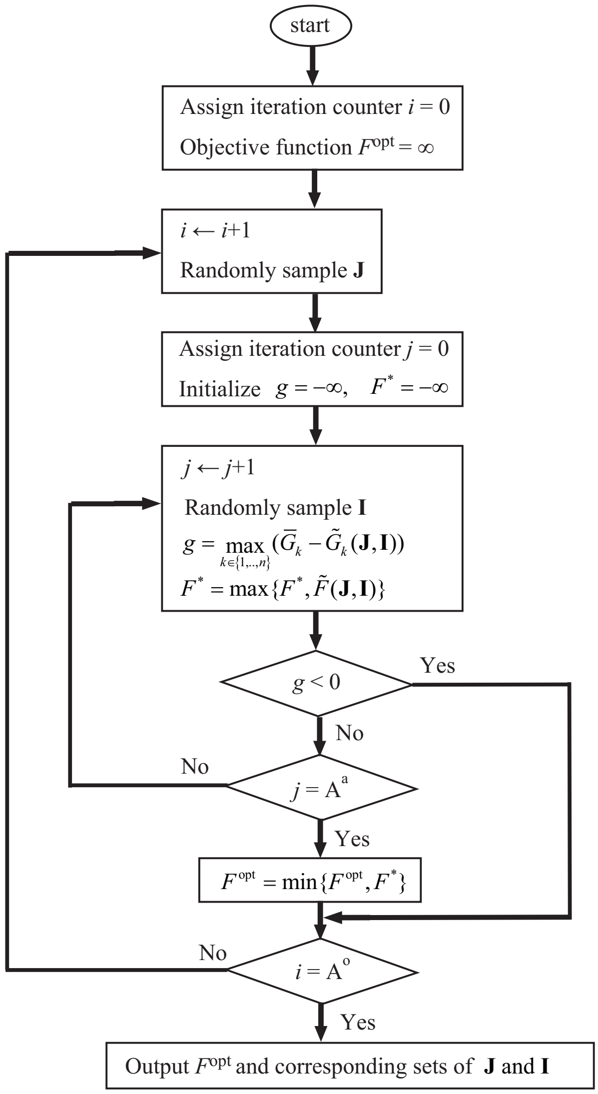

Fig. 3 Algorithm of RS for a two-stage problem of optimization and anti-optimization.

We can evaluate the objective functions of H samples and find the value F^* of the $\tilde{f}H$th worst solution; i.e., $F^* = X(1-\tilde{f})$, where X is the quantile function in the sampled set. Let f denote the probability of exceeding F^* in the original set. Then, the following inequality holds from the Chernoff bound (Mitzenmacher and Upfal, 2005):

$$\Pr(f \notin [\tilde{f} - \delta, \tilde{f} + \delta]) < \mathrm{e}^{-H\delta^2/(2f)} + \mathrm{e}^{-H\delta^2/(3f)} \tag{7}$$

where the unknown probability f should be estimated. The most conservative value for f is 1, while a tighter upper bound can be given, if a smaller upper bound is available for f. Note that several alternative forms exist for the Chernoff bound.

3.3 Genetic algorithms

We also carry out genetic algorithm (GA) for comparison purpose. The details of GA are not presented here, because there exist many text books, e.g., (Goldberg, 1989). The ranking strategy in Ohsaki (1995) and the elitist strategy are used with the framework of simple GA. Note that the purpose of this paper is to present an efficient method for structural optimization problem, where computational cost for each function evaluation is very large and we cannot carry out analyses many times. Therefore, small numbers are given for population size and number of generations for GA.

4 Mathematical example

4.1 Problem formulation

We first compare the performances of TS, GA, and RS using a small mathematical optimization problem. Let $-4 \leq x_i \leq 4$ $(i = 1, \dots, 4)$ denote the variables. The uncertain parameters are denoted by $-1 \leq p_i \leq 1$ $(i = 1, \dots, 4)$. The objective function is given as

$$\begin{aligned} \tilde{F}(\mathbf{x}, \mathbf{p}) = & \sum_{i=1}^{4} (x_i^4 - 16x_i^2 + 5x_i) \\ & + a[(p_1^2 + 0.1p_1)x_1 + (p_2^2 + 0.2p_2)x_2 \\ & + p_3 x_3 + 0.9 p_4 x_4] \end{aligned} \tag{8}$$

where $a = 10$ is a specified parameter, $\mathbf{x} = (x_1, \dots, x_4)$, and $\mathbf{p} = (p_1, \dots, p_4)$. This function without parameter uncertainty ($a = 0$) has 2^n local minima; hence, it is called $2n$-minima function (Yasudada *et al.*, 2008). Fig. 4 shows the contour lines for $n = 2$ with $a = 0$. The optimal solution exists at $(x_1, x_2) = (-2.9035, -2.9035)$, and local minima are found at $(2.7468, -2.9035)$, $(-2.9035, 2.7468)$, and $(2.7468, 2.7468)$.

The upper-level optimization problem is written as

$$\text{Minimize} \quad \hat{F}(\mathbf{x}) \tag{9a}$$
$$\text{subject to} \quad -4 \leq x_i \leq 4, \quad (i = 1, \dots, 4) \tag{9b}$$

where the worst value $\hat{F}(\mathbf{x})$ is obtained by solving the following anti-optimization problem:

$$\text{Find} \quad \hat{F}(\mathbf{x}) = \max_{\mathbf{p}} \tilde{F}(\mathbf{x}, \mathbf{p}) \tag{10a}$$
$$\text{subject to} \quad -1 \leq p_i \leq 1, \quad (i = 1, \dots, 4) \tag{10b}$$

The variables and parameters are defined using integer values as follows:

$$x_i = -4 + \Delta x(J_i - 0.5), \quad (i = 1, \dots, 4) \tag{11a}$$
$$p_i = -1 + \Delta p(I_i - 0.5), \quad (i = 1, \dots, 4) \tag{11b}$$

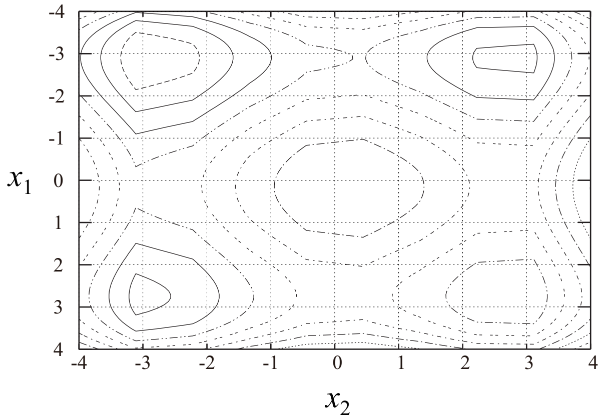

Fig. 4 Contour lines of $2n$-minima function for $n = 2$ without parameter uncertainty.

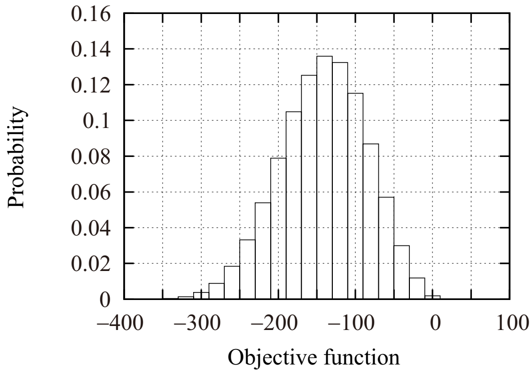

Fig. 5 Probability of objective function by enumeration.

where

$$\Delta x = 8/(s - 1), \quad J_i \in \{1, \dots, s\} \tag{12a}$$
$$\Delta p = 2/(q - 1), \quad I_i \in \{1, \dots, q\} \tag{12b}$$

Finally, the hybrid problem with integer variables $\mathbf{J} = (J_1, \dots, J_4)$ and integer parameters $\mathbf{I} = (I_1, \dots, I_4)$ is formulated as

$$\text{Minimize} \quad \hat{F}(\mathbf{J}) = \max_{\mathbf{I}} \tilde{F}(\mathbf{J}, \mathbf{I}) \tag{13a}$$
$$\text{subject to} \quad J_i \in \{1, \dots, s\}, \quad (i = 1, \dots, 4) \tag{13b}$$
$$I_i \in \{1, \dots, q\}, \quad (i = 1, \dots, 4) \tag{13c}$$

4.2 Performance of random sampling for anti-optimization

The performance of RS for anti-optimization is compared with those of TS and GA. A rather large value 8 is assigned for s and q for the convenience in coding of GA.

Table 1 Values of $R_i^{\min}$ of 50 solutions for five cases with different random seeds by RS, GA, and TS.

(a) RS

Case	1	2	3	4	5	Average
Maximum	90	110	128	118	120	113.2
Minimum	1	1	1	1	2	3
Average	31.780	25.120	22.760	29.560	29.240	27.692
Standard deviation	25.224	24.756	25.109	28.768	25.920	25.956

(b) GA

Case	1	2	3	4	5	Average
Maximum	490	357	428	273	67	323.0
Minimum	1	1	1	1	1	1
Average	54.640	32.660	32.120	26.000	15.200	32.124
Standard deviation	91.058	67.587	68.444	48.683	18.123	58.779

(c) TS

Case	1	2	3	4	5	Average
Maximum	40	210	70	74	45	87.8
Minimum	1	1	1	1	1	1
Average	10.640	15.300	13.600	13.440	14.560	13.508
Standard deviation	8.390	30.550	13.454	15.170	11.632	15.839

Therefore, we have $8^4 = 4096$ solutions and 4096 parameter sets for each solution; hence, the total number of different solution-parameter sets is $8^8 = 16777216$. The objective values of all 16777216 sets are enumerated to obtain the probability of objective function as shown in Fig. 5.

In order to evaluate the performance of RS, the worst value is found from the 144 sets, which is 3.52% of the total 4096 sets, of randomly sampled parameters for each of 50 solutions. As discussed in the previous section, 50 trials are enough for a good approximation of the worst solution; however, for comparison purpose with the genetic algorithm, a rather larger 144 trials are used also for RS and TS.

Let R_{ij} $(i = 1, \ldots, 50; j = 1, \ldots, 144)$ denote the order of the jth parameter set, which is assigned in non-increasing order of the objective function, among the 4096 parameter sets for the ith solution. The smallest (worst) order of the parameter sets for each solution is denoted by $R_i^{\min}$; i.e.,

$$R_i^{\min} = \min_j R_{ij}, \quad (i = 1, \ldots, 50) \tag{14}$$

Table 1(a) shows the maximum, minimum, and average values, as well as the standard deviation of $R_i^{\min}$ among the 50 solutions for five cases denoted by Cases 1–5 with different initial random seeds. The average values among five cases are also listed in the last column. As is seen, the average value of $R_i^{\min}$ is 27.692, which is sufficiently small compared with the total number 4096 of the parameter sets. Furthermore, the worst parameter set, i.e., $R_{ij} = 1$, is obtained in at least one of the 50 solutions for three cases. The maximum value is 128, which is 3.13% of the 4096 parameter sets.

GA is carried out for comparison purpose. We use the ranking strategy in Ohsaki (1995) and elitist strategy. The probabilities of single-point crossover and mutation are 0.8 and 0.01, respectively, and the number of elite solutions is 2. The numbers of population and generation are 12; i.e., the total number of function evaluation is 144, which is same as RS. The results are listed in Table 1(b). As is seen, the worst parameter set is found for at least one of 50 solutions; however, the maximum order is 490, which is very large, and the average order also has larger value than that of RS.

Similar investigation is also carried out for TS to obtain the results in Table 1(c), which shows that TS is superior to RS in average performance; however, TS sometimes has very poor performance and the maximum value has larger order than that of RS. Therefore, RS is a very effective approach for optimization problems that have several local optima, because no problem-dependent parameter exists for RS.

For solving a hybrid problem of optimization and anti-optimization in next section, we further decrease the numbers of sampling values and function evaluations. Therefore, the performances of RS, TS, and GA for small number 5 of the sampling parameter value q is investigates; i.e., we have $5^4 = 625$ parameter sets for each solution.

Tables 2(a), (b), and (c) for RS, GA, and TS, respectively, show the maximum, minimum, and average values, as well as the standard deviation of $R_i^{\min}$ among the 50 with different numbers of analyses. As is seen, the average value and standard deviation decrease as the number of selections is increased, although the maximum value sometimes increases. It is seen from these

Table 2 Values of $R_i^{\min}$ of 50 solutions with different numbers of analyses by RS, GA, and TS.

(a) RS

Number of selections	Maximum	Minimum	Average	Std. dev.
50	46.4	1	9.116	10.399
60	35.4	1	7.180	8.326
70	26.6	1	6.720	6.975
80	31.8	1	6.072	7.330
90	31.2	1	5.612	6.602
100	26.6	1	5.152	6.318

(b) GA

Population	generation	Maximum	Minimum	Average	Std. dev.
7	7	172.0	1	23.372	37.689
8	8	249.0	1	25.652	47.280
9	9	130.0	1	12.056	23.044
10	10	195.2	1	14.888	34.075
11	11	83.0	1	6.472	14.631

(c) TS

$N^{\rm b}$	$N^{\rm s}$	Maximum	Minimum	Average	Std. dev.
5	10	84.8	1	7.124	14.451
6	10	65.6	1	5.984	11.406
6	11	55.0	1	4.924	9.396
7	11	55.0	1	3.776	9.196
7	12	43.0	1	4.184	7.826
8	12	39.0	1	4.328	7.450

results that TS has smallest average values; however, the maximum values are larger than those of RS.

4.3 Performance of random sampling for hybrid problem

It has been shown in Sec. 4.2 that the performance of GA is not always good, if the number of function evaluations is limited to be very small. Therefore, we compare the performances of RS and TS for the hybrid problem. The numbers of sampling values of variables and parameters are given as $s = 5$ and $q = 5$. Therefore, we have $5^4 = 625$ solutions and 625 parameter sets for each solution; hence, the total number of different solution-parameter sets is $5^8 = 390625$.

Since good approximate worst solution is found with 50 selections, we fix the number of selections as 50 for both of upper and lower problems. The approximate optimal solutions found by RS are listed for five cases with different random seeds in Table 3, where 'objective value $F^{\rm A}$' denotes the value of objective function corresponding to the approximate worst parameter set $\mathbf{I}^{\rm A}$ of the approximate optimal solution $\mathbf{J}^{\rm A}$, and 'worst value $F^{\rm W}$' denotes the exact worst objective value of the approximate optimal solution; i.e.,

$$\begin{aligned} F^{\rm A} &= \tilde{F}(\mathbf{J}^{\rm A}, \mathbf{I}^{\rm A}), \\ F^{\rm W} &= \max_{\mathbf{I}} \tilde{F}(\mathbf{J}^{\rm A}, \mathbf{I}) \end{aligned} \qquad (15)$$

The order of $F^{\rm A}$ in the complete list of 625 parameter sets by enumeration for each solution is denoted by 'order'. Note that the objective values cannot be compared among different cases, because they have different approximate solutions.

The optimal solution by enumeration is $\mathbf{J} = (1, 1, 1, 1)$ with the worst objective value -251.29 corresponding to the parameters $\mathbf{I} = (3, 3, 1, 1)$. Although the exact solution has not been found within five trials, a good approximate solutions has been found using RS.

The results of different numbers of neighborhood solutions $N^{\rm b}$ and steps $N^{\rm s}$ are listed in Table 4, where the length of tabu list is $N^{\rm t} = N^{\rm b}N^{\rm s}$. Note that the results do not depend strongly on $N^{\rm b}$ and $N^{\rm s}$, if the total number of analyses is almost the same.

Finally, a quantitative evaluation is carried out for RS using larger number of solutions with $q = 20$. The variables J_i $(i = 1, \dots, 4)$ are fixed at 1, and only the second term in (8) corresponding to parameter uncertainty is considered; therefore, the total number of combination is $20^4 = 160000$. Random selection is carried out 10000 times from five different random seeds. The histories of order of worst solutions are plotted with respect to the number of selections in Fig. 6 in logarithmic scale. As is seen, the solution within 1600th worst corresponding to the accuracy of 0.01 can be found with less than 500 selections for all five cases.

Table 3 Approximate optimal solutions by RS.

	Variables				Parameters				Objective value F^{A}	Worst value F^{W}	Order of F^{A}
Case 1	1	1	1	5	2	1	1	5	−238.49	−219.29	20
Case 2	1	1	5	1	4	3	5	2	−237.21	−219.29	17
Case 3	1	2	2	1	3	3	1	1	−198.94	−198.94	1
Case 4	1	1	5	1	4	3	5	2	−237.21	−219.29	17
Case 5	1	4	2	1	2	5	3	1	−186.77	−170.14	22

Table 4 Approximate optimal objective values by TS.

N^{b}	N^{s}	Case 1	Case 2	Case 3	Case 4	Case 5
4	12	−221.25	−221.25	−199.06	−176.25	−221.25
5	10	−253.75	−221.25	−148.75	−180.00	−196.25
6	8	−243.75	−191.25	−207.50	−176.25	−168.75

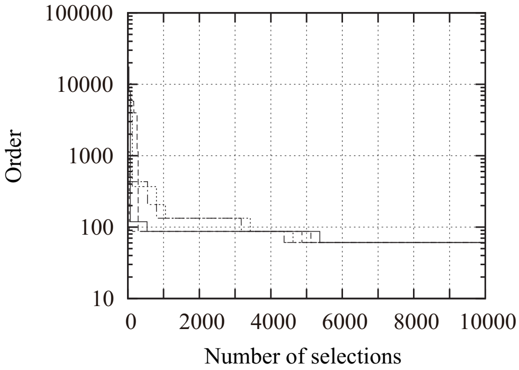

Fig. 6 History of order of worst solution for five cases with $q = 20$.

4.4 Probabilistic analysis

Suppose we carry out probabilistic analysis to find the objective value of 4% exceedance. The objective values F_i $(i = 1, \dots, 5)$ of the 400th worst solutions in 10000 samples for five cases are 22.35, 22.15, 22.15, 22.35, 22.15; i.e., the quantile function for the probability 0.96 $(= 1 - 0.04)$ is 22.35, 22.15, 22.15, 22.35, and 22.15, respectively, for five cases. Let f_i denote the probability in the original set to exceed these five values, respectively. Then, using the Chernoff bound (7) assuming 1% confidence, probability for $|f_i - 0.04| \leq 0.01$ are bounded by 0.042412 if we assume $f_i \leq 0.1$, and by 0.0013180 if $f_i \leq 0.05$. In fact, the orders of F_i in the complete list of 160000 solutions are 6361, 6685, 6685, 6361, 6685; therefore, the probabilities for exceeding F_i are 0.039756, 0.041781, 0.041781, 0.039756, 0.041781, which are very close to 0.04.

5 Optimization of building frame

5.1 Description of design problem

Optimal cross-sections are found for a 4-story single-span plane steel frame model as shown in Fig. 7 subjected to severe seismic motions. The same notations as Sec. 4 are used for the variables and parameters. The objective function is the total structural volume $V(\mathbf{J}, \mathbf{I})$. The steel sections of beams and columns are selected from the list of standard sections in Table 5, where 'H' means wide-flange section, and 'HSS' means tubular hollow square section.

The design is designated by the section numbers in the list. Beams are classified into two groups denoted by Beam 1 in 2nd and 3rd floors and Beam 2 in 4th floor and roof, which are defined by the variables J_1 and J_2, respectively. Columns are also classified into two groups denoted by Columns 1 and 2, respectively, consisting of those in 1st and 2nd stories and those in 3rd and 4th stories, which are defined by the variables J_3 and J_4, respectively. Thus, we have four design variables. The sections of all columns are selected from the same list, because the external size of all columns should be the same for this low-rise building frame. The standard model consisting of H-400×200×9×19 for Beam 1, H-350×175×7×11 for Beam 2, and HSS-300×300×16 for Columns 1 and 2 is designated by $\mathbf{J} = (3, 3, 3, 3)$.

The steel material has a bilinear stress-strain relation, where Young's modulus is 205 kN/mm^2, the nominal value of hardening ratio of linear kinematic hardening is 0.01, and the nominal values of yield stresses of beams and columns are 235 N/mm^2.

The artificial seismic motions are generated using the standard superposition method of sinusoidal waves (Iyengar and Rao, 1979). The target acceleration spectrum is the design acceleration response spectrum for 5% damping specified by Notification 1461 of the Min-

Table 5 List of standard sections for beams and columns.

	Beam 1 (2F, 3F)	Beam 2 (4F, RF)	Column
1	H-400×200×9×12	H-250×125×6×9	HSS-300×300×9
2	H-400×200×9×16	H-300×150×6.5×9	HSS-300×300×12
3	H-400×200×9×19	H-350×175×7×11	HSS-300×300×16
4	H-400×200×9×22	H-400×200×8×13	HSS-300×300×19
5	H-400×200×12×22	H-450×200×9×14	HSS-300×300×22

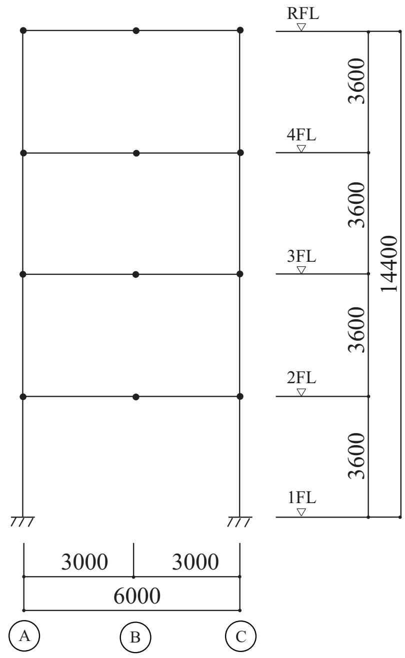

Fig. 7 A 4-story plane frame.

istry of Land, Infrastructure and Transport (MLIT), Japan. Although the details of parameters used in building engineering are not presented here, the acceleration spectrum S_a is given as a function of the natural period T as

$$\begin{aligned} S_\mathrm{a}(T) &= (0.96 + 9T)C && \text{for } T \leq 0.16 \\ S_\mathrm{a}(T) &= 2.40C && \text{for } 0.16 \leq T \leq 0.64 \\ S_\mathrm{a}(T) &= 1.536C/T && \text{for } 0.64 \leq T \end{aligned} \tag{16}$$

where C is the scaling factor, which is 7.5 in the following example. The seismic motion with duration 20 sec. is applied at the base of the frame in horizontal direction. A constraint is given for the maximum value among the mean-maximum interstory drifts of all stories against five artificial motions, which is simply denoted by *maximum interstory drift* $D^\mathrm{m}(\mathbf{J}, \mathbf{I})$. By assigning an appropriately small upper bound for D^m, the frame does not collapse under severe earthquakes.

A general purpose frame analysis software called OpenSees (PEERC, 2006) is used for seismic response analysis of the frame. Each column is modeled by a beam-column element, whereas each beam is divided into two elements. The sections of elements are divided into fibers. Since only plane frame analysis is carried out, the flange and web of the beam are discretized into 4 and 16 fibers only in the directions of thickness and depth, respectively. The integration is carried out using the Gauss-Lobatto rules that has integration points at the ends of the element, where the number of integration points is 8; thus, the plastification at the member ends can be accurately detected. The standard Newmark-β method ($\beta = 0.25, \gamma = 0.5$) is used for integration in time domain with the increment of 0.01 sec.

The stiffness-proportional damping is used with the damping ratio 0.02 for the first mode. The fundamental period of the standard design with $\mathbf{J} = (3, 3, 3, 3)$ is 0.71 sec, which means from (16) that the response acceleration reduces as the natural period becomes larger as the result of plastification.

5.2 Anti-optimization of standard design

We first carried out preliminary parametric study to investigate the effect of various parameters on the maximum interstory drift D^m, and found that D^m is a monotonically decreasing function of the yield stress of columns and the hardening ratios of beams and columns. Therefore, the smallest possible values should be chosen for these parameters to obtain the worst response; hence, the yield stress of column has the lower bound value 235 N/mm^2, and the hardening coefficient is equal to the specified small value 0.01.

In contrast, D^m is not a monotonic function of the yield stress $\sigma_\mathrm{Y}^\mathrm{b}$ of beams. The range of uncertainty is given as 20 % of the nominal value, which is discretized into five equally spaced values. Note that the nominal value 235 N/mm^2 of yield stress indicates the lower bound according to the specification of steel material. Therefore, the integer parameters I_1 and I_2 for $\sigma_\mathrm{Y}^\mathrm{b}$ of Beams 1 and 2, respectively, correspond to 239.7, 249.1, 258.5, 267.9, and 277.3 N/mm^2 for the integer values 1, 2, 3, 4, and 5.

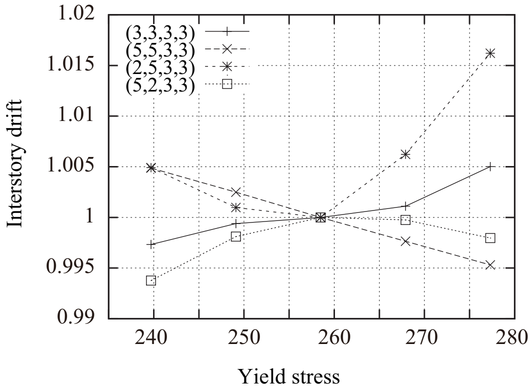

Fig. 8 Relation between yield stress of beams and normalized values of maximum interstory drifts of various designs.

Fig. 8 shows the values of D^{m} normalized by those for $\sigma_{\mathrm{Y}}^{\mathrm{b}} = 258.9$. For the standard design $\mathbf{J} = (3, 3, 3, 3)$, D^{m} is an increasing function of $\sigma_{\mathrm{Y}}^{\mathrm{b}}$, because a strong beam leads to a column-collapse mechanism that has small energy dissipation and large local interstory drift. For the design (5,5,3,3) with stronger beam, D^{m} is an decreasing function of $\sigma_{\mathrm{Y}}^{\mathrm{b}}$, because, in this case, a larger $\sigma_{\mathrm{Y}}^{\mathrm{b}}$ leads to larger energy dissipation without changing the collapse mechanism. Finally, the values of D^{m} for intermediate designs (2,5,3,3) and (5,2,3,3) are not monotonic functions of $\sigma_{\mathrm{Y}}^{\mathrm{b}}$.

In addition to material parameters, the cross-sectional geometry also has uncertainty. We consider uncertainty in the thicknesses of flanges of Beams 1 and 2, defined by I_3 and I_4, respectively, and assume that their nominal values are mean values. The ranges of uncertainty are given as 10 % of the nominal values, which are discretized into five equally spaced values; hence, the possible values are determined by multiplying 0.96, 0.98, 1.0, 1.02, and 1.04 to the nominal value. Using these notations, the anti-optimization problem is formulated as

$$\text{Find} \quad \hat{D}^{\mathrm{m}}(\mathbf{J}) = \max_{\mathbf{I}} \tilde{D}^{\mathrm{m}}(\mathbf{J}, \mathbf{I}) \tag{17a}$$

$$\text{subject to} \quad I_i \in \{1, \dots, 5\}, \quad (i = 1, \dots, 4) \tag{17b}$$

Since we consider uncertainty in four parameter values, the number of possible combinations of parameters is $5^4 = 625$. Although the distribution of each parameter can be modeled parametrically using, e.g., normal distribution, we use the uniform distribution, because our purpose is to find good approximate worst solutions. Fig. 9 shows the discretized probability density function of D^{m} for the 625 parameter sets obtained by enumeration for the standard design. The maximum and minimum values are 0.0518 m and 0.0475 m, respectively.

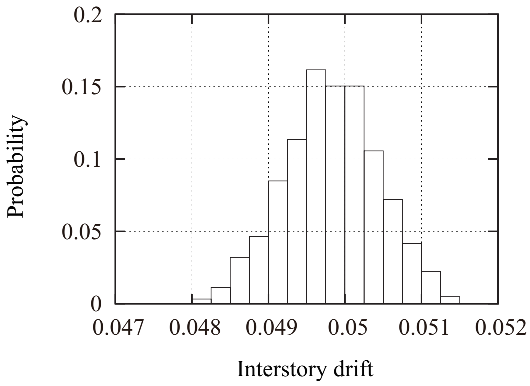

Fig. 9 Probability density function of D^{m} by enumeration.

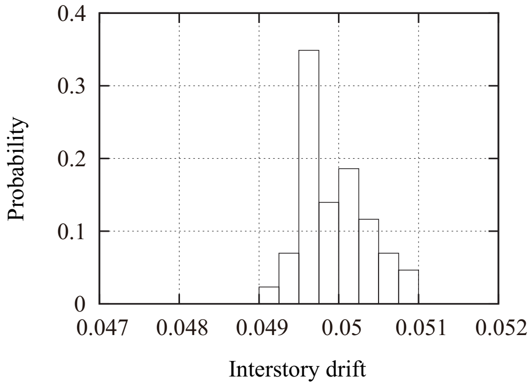

Fig. 10 Probability density function of D^{m} by TS.

We assume the parameter sets corresponding to D^{m} up to the 50th maximum value are regarded as approximate worst parameter sets, and select 50 parameter sets randomly from 625 samples. Then, the probability that no approximate worst parameter set is found through this random sampling is $(575/625)^{50} = 0.0154$, which is very small. For TS, the number of neighborhood solutions N^{b} is 5, and the number of steps N^{s} is 10; hence, the number of analyses is also 50 and the length of tabu list is 50. We carry out RS and TS five times starting with different initial random seeds for comparison purpose.

The maximum value, minimum value, mean value, and the standard deviation of D^{m} obtained by TS is listed in Table 6(a), where the second row is the order of the worst (minimum) value in the original list of 625 parameter sets. As is seen, the worst parameter set has been found for four cases; however, the 37th worst set has been found for Case 1. The standard deviation is less than 1/10 of the difference between the maximum and minimum values in the original list. Fig. 10 shows the probability density of 50 solutions for a single run of TS (Case 5). As seen from Figs. 9 and 10, TS searches the solutions with larger responses. This way, TS mostly

Table 6 Values of D^{m} by TS and RS for anti-optimization of the standard design with five different random seeds.

(a) TS

Case	1	2	3	4	5
Maximum	0.050811	0.051322	0.051342	0.051350	0.051350
Order of maximum	37	1	1	1	1
Minimum	0.049082	0.048551	0.049743	0.048701	0.049334
Average	0.049936	0.050354	0.050680	0.050310	0.050654
Standard deviation (10^{-4})	4.0499	7.3122	3.7777	7.0951	5.2047

(b) RS

Case	1	2	3	4	5
Maximum	0.051164	0.051164	0.051121	0.051240	0.051063
Order of maximum	7	7	9	4	12
Minimum	0.048345	0.048082	0.048680	0.048345	0.048345
Order of minimum	621	625	605	621	621
Average	0.049623	0.049764	0.049874	0.049701	0.049817
Standard deviation (10^{-4})	6.2438	7.2518	6.5434	6.5169	6.2771

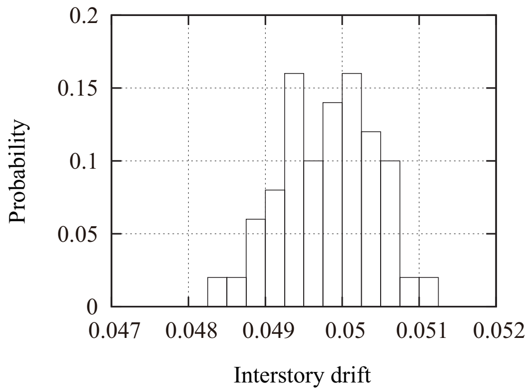

Fig. 11 Probability density function of D^{m} by RS.

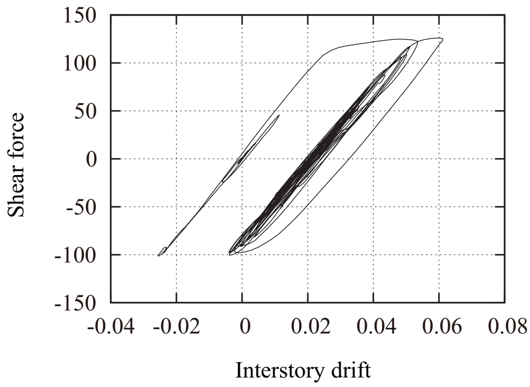

Fig. 12 Relation between interstory drift and shear force of 1st story.

has a good performance through analyses of less than 1/10 of the size of the original list, but sometimes fails to obtain a good approximate solution, although the 37th worst solution can be regarded as an approximate worst solution in our definition.

Table 6(b) shows the results of RS, in which the maximum order varies between 4th and 12th. Therefore, the average performance of RS is better than that of TS, although the global worst solution could not be found by RS. This way, RS can find a good approximate worst solution within the analyses of 1/10 of the total number of solutions. The minimum value and its order in the original list is also listed in Table 6(b) to show the range of variation of response due to the parameter uncertainty. Fig. 11 shows the probability density of 50 solutions for a single run of RS, which is similar to Fig. 9.

5.3 Two-stage problem of optimization and anti-optimization

In the upper-level optimization problem, we select member sections from the pre-assigned list of standard sections in Table 5. A constraint is given such that the worst value of $\tilde{D}^{\mathrm{m}}(\mathbf{J}, \mathbf{I})$ is not more than $\bar{D}^{\mathrm{m}} = 0.072$ m, which is equivalent to 2% of the interstory drift angle. The structural optimization problem for minimizing the total structural volume $\hat{V}(\mathbf{J})$ considering parameter uncertainty is formulated as

$$\text{Minimize} \quad \hat{V}(\mathbf{J}) = \max_{\mathbf{I}} \tilde{V}(\mathbf{J}, \mathbf{I}) \tag{18a}$$

$$\text{subject to} \quad \tilde{D}^{\mathrm{m}}(\mathbf{J}, \mathbf{I}) \leq \bar{D}^{\mathrm{m}} \tag{18b}$$

$$J_i \in \{1, \dots, 5\}, \quad (i = 1, \dots, 4) \tag{18c}$$

$$I_i \in \{1, \dots, 5\}, \quad (i = 1, \dots, 4) \tag{18d}$$

The performances of RS and TS are compared, where $N^{\mathrm{b}} = 5$, $N^{\mathrm{s}} = 10$, and the length of tabu list is 50 for

Table 7 Approximate optimal solutions by RS and TS.

(a) RS

	Variables **J**				Parameters **I**				Objective value	Interstory drift	Order of objective value
Case 1	3	3	1	1	5	4	1	1	0.49549	0.052664	10
Case 2	4	2	1	1	3	1	2	1	0.48993	0.069918	9
Case 3	1	2	1	1	5	2	1	1	0.44407	0.068477	1
Case 4	1	5	1	1	2	2	1	2	0.50245	0.059975	11
Case 5	2	2	2	2	5	1	5	3	0.55103	0.067830	41

(b) TS

	Variables **J**				Parameters **I**				Objective value	Interstory drift	Order of objective value
Case 1	1	2	1	1	3	1	1	1	0.44407	0.068381	1
Case 2	1	2	1	1	5	1	1	1	0.44407	0.068673	1
Case 3	1	2	1	1	3	1	1	1	0.44407	0.068349	1
Case 4	1	3	2	1	1	3	1	5	0.50778	0.058050	15
Case 5	1	2	5	1	5	5	3	2	0.63127	0.068346	125

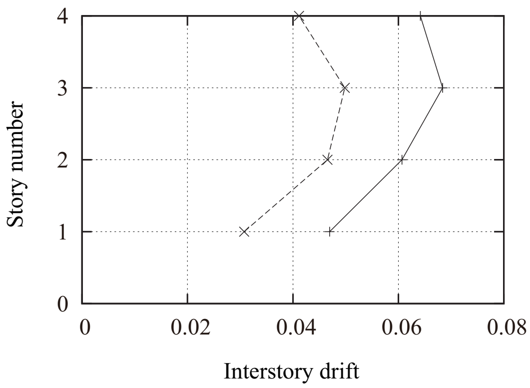

Fig. 13 Envelop of maximum interstory drifts.

Table 8 Global optimal solution $\mathbf{J} = (1, 2, 1, 1)$ by TS.

Beam 1 (2F, 3F)	H-400×200×9×12
Beam 2 (4F, RF)	H-300×150×6.5×9
Column 1 (1S, 2S)	HSS-300x300x9
Column 2 (3S, 4S)	HSS-300x300x9
Yield stress:	
Beam 1	239.7 N/mm^2
Beam 2	249.1 N/mm^2
Thickness of flange:	
Beam 1	11.52 mm
Beam 2	8.64 mm
Objective function	0.44407
Max. story drift	0.068381

TS of both of the optimization and anti-optimization problems. The solutions violating the constraint is simply rejected in RS and TS. The results by RS and TS are listed in Tables 7(a) and (b), respectively. Among five trials of RS, the global optimal solution has been found once, and the maximum order is 41st, which can be regarded as an approximate solution within the 50th. TS may sometimes lead to a very bad solution as Case 5 in Table 7(b), although the global optimal solution is found in three trials,

The relation between the interstory drift and shear force of the 1st story is plotted in Fig. 12 for the optimal design with worst parameter set subjected to one of the five seismic motions. As is seen, residual story drift exists due to plastification of the frame. The envelope of maximum interstory drifts is plotted in solid line in Fig. 13. The dotted line is the result of the standard frame model with the parameter set $\mathbf{I} = (3, 3, 3, 3)$. The interstory drifts increase to slightly smaller value than the upper bound as a result of optimization.

6 Conclusions

A random sampling approach has been presented for a two-stage problem of optimization and anti-optimization for structural design, which is also called worst-case design problem. A new concept is introduced for the anti-optimization problem, where the accuracy of the solution is defined by the order rather than the value of the response among the possible combinations of the parameters. Hence, the approximate solutions for the optimization and anti-optimization problems are found using a heuristic approach, including TS and RS, for combinatorial problems.

Optimal cross-sections have been found for a steel building frame considering uncertainty of parameters for material and cross-sectional geometry. The objective function is the total structural volume, and a constraint is given for the worst value of the interstory drifts under a set of seismic motions. The cross-sections are selected from the list of available standard sections. The parameters are also discretized into integer values. Therefore, the optimum design problem and the anti-optimization problem for finding the worst response are formulated as combinatorial problems. It has been shown that a good approximate solution can be found using RS within a small number of analyses, which is less than 1/10 of the total number of solutions. In contrast, TS has moderately good performance, but sometimes leads to a very bad solution.

The performance of RS has also been compared with TS and GA using a small mathematical problem. The conclusions drawn from these numerical experiments are as follows:

1. TS and GA are better than RS in view of accuracy, because they can find the optimal/anti-optimal solution in most of the trials. However, TS and GA may sometimes result in very bad solutions. In contrast, RS has a good average performance.
2. RS does not have any problem-dependent parameter that should be assigned by intuition, which is regarded as a very important superiority to other optimization algorithms. Furthermore, the required number of trials in RS can be estimated rigorously using the probability of failing to obtain an approximate solution defined by the order of objective value. Therefore, RS can be carried out only once, while other *efficient* algorithms need many preliminary analyses for parameter tuning.
3. If analysis can be carried out many times, the accuracy of solution can be investigated in a probabilistic manner using the Chernoff bound.
4. The performance of RS is related to the ratio of number of approximate solutions to the number of total solutions, and does not depend on the number of variables/parameters directly.

It should be noted that the purpose of this paper is to investigate the performance of RS for finding approximate solutions to structural optimization problems with limited small number of analyses. It is true that RS has poor performance when the global optimal or anti-optimal solution should be obtained, and GA is superior to RS, if the functions can be evaluated many times, e.g., more than 10000 times. However, we need only an approximate solution with less than 100 analyses. RS is also effective for problems with several local optimal solutions. The proposed method can be effectively applied to practical design problems, where the bounds of parameters cannot be given rigorously.

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