Imperfection Sensitivity of Degenerate Hilltop Branching Points

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Abstract

Imperfection sensitivity is investigated for a degenerate hilltop branching point, where a degenerate bifurcation point exists at a limit point. A degenerate hilltop branching point is important as it is a byproduct of optimization of shallow shell structures under nonlinear buckling constraints. A systematic procedure is presented for asymptotic sensitivity analysis based on enumeration of vertices of a convex region defined by linear inequality constraints on the orders of the variables. The effectiveness of the proposed method is demonstrated by sensitivity analysis of degenerate hilltop branching points, considering minor and major imperfections, corresponding to an unstable-symmetric or asymmetric bifurcation point at the limit point. It is found that a hilltop branching point can be imperfection sensitive.

Key words: Asymptotic analysis, Degenerate critical point, Hilltop branching, Imperfection sensitivity, Vertex enumeration

1 Introduction

It is well known that imperfection sensitivity of nonlinear buckling load of an elastic conservative system is enhanced as a result of coincidence of critical points [1, 2]. The imperfection sensitivity is severe for the semi-symmetric bifurcation point, where unstable-symmetric and asymmetric bifurcation points coincide. A coincident critical point that has a bifurcation point at a limit point is called a hilltop branching point, or simply *a hilltop point*, which was observed in mechanical instability of stressed atomic crystal lattices [3, 4], steel specimens [5], and structural models [6, 7]. Contrary to coincident bifurcation points, a hilltop point has less severe piecewise linear law of imperfection sensitivity [8, 9].

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It is known that optimization of a symmetric structure under constraints on nonlinear buckling load factor often engenders a coincident critical point [10]. For a symmetric structure, an imperfection with some symmetry corresponds to a minor (second-order) imperfection, for which the imperfection sensitivity is usually less severe than an asymmetric major (first-order) imperfection. However, it has been pointed out by the first author that the minor imperfection sometimes dominates over the major imperfection [11, 12]. For shallow shell-type structures, the optimal solution often is achieved at a hilltop branching point [6]. Therefore, imperfection sensitivity of the hilltop branching point has been of practical interest [7, 9, 13–15].

A degenerate critical point is characterized by vanishing of the derivative of the corresponding eigenvalue of the tangent stiffness matrix along the fundamental equilibrium path. Sensitivity coefficients of a degenerate bifurcation point were investigated by Ohsaki [16]. However, a degenerate critical point has somewhat been set aside as a rare exceptional case that can arise from accidental vanishing of some differential coefficients of the potential function. Moreover, much care is not paid to this critical point in stability design, because the point has been believed to have no negative effect on the stability of the structure; there is no bifurcation path, and the equilibrium path of an imperfect system makes only a slight detour around the point [16]. However, it has been demonstrated in Ref. [6] that the optimal solution of a shallow truss may have a degenerate hilltop point, which has bifurcation paths and enhances the imperfection sensitivity as a result of vanishing of some differential coefficients of the potential function.

Characteristics of critical points of imperfect systems have been investigated by asymptotic approaches assuming that the imperfection parameters are sufficiently small, e.g. [1, 17]. The asymptotic approaches are applied to imperfection sensitivity analysis of structures of various types, e.g. [18, 19]. It is rather easy to evaluate in an *ad hoc* manner the orders of the load factor and generalized coordinates at a simple critical point with respect to the imperfection parameter. However, it is very difficult to find consistent orders intuitively from the bifurcation equations and the criticality condition of coincident critical points. Moreover, there is no guarantee that all possible combinations of the orders have been exhausted. In this paper, imperfection sensitivity is investigated for a degenerate hilltop point based on asymptotic analysis. A systematic approach to enumeration of the consistent set of orders of the variables corresponding to the coincident critical points, especially for degenerate hilltop points, is presented. Although the Newton Polygon [20] is an established mathematical tool, we present a simple approach consistent with symbolic computation and numerical enumeration method. Minor and major imperfections are considered for unstable-symmetric and asymmetric bifurcation points at the limit point. It is pointed out that the degenerate hilltop points often are imperfection sensitive, unlike the nondegenerate hilltop points that enjoy the piecewise-linear law and are not imperfection-sensitive.

2 Asymptotic formulation of critical points

We follow a standard formulation of nonlinear bifurcation analysis [9] and offer its brief summary.

Consider a finite-dimensional elastic conservative system that is subjected to a set of loads parameterized by the load factor Λ. The vector of generalized displacements, which defines the nodal locations after deformation, is denoted by $\mathbf{u} = (u_1, \dots, u_n)^\top$, where n is the number of degrees of freedom of the structure. The equilibrium equation is defined by the stationary condition of the total potential energy $\widehat{\Pi}(\mathbf{u}, \Lambda)$ as

$$\frac{\partial \widehat{\Pi}}{\partial u_i} = 0, \quad (i = 1, \dots, n) \tag{1}$$

The criticality condition is given by using the stability matrix (tangent stiffness matrix) $\mathbf{S}$ as

$$\det \mathbf{S} = 0 \tag{2}$$

where the (i, j) component S_{ij} of $\mathbf{S}$ reads

$$S_{ij} = \frac{\partial^2 \widehat{\Pi}}{\partial u_i \partial u_j}, \quad (i, j = 1, \dots, n) \tag{3}$$

For a structure with a large number of degrees of freedom n, the nonlinear governing equations (1) and (2) defining the critical point involve a large number of independent variables and nonlinear terms and, hence, are highly complex. In *general theory of nonlinear stability* [21], to obtain asymptotic general forms of imperfection sensitivity laws, the nonlinear governing equations were simplified on the basis of the following two steps:

(1) The equilibrium equation is reduced to the *bifurcation equation* with only a few active independent variables by the elimination of passive coordinates [22].

(2) Higher-order terms of the bifurcation equation are truncated by an asymptotic assumption.

A formulation based on this reduction is herein called V-formulation [1, 23–27]. The formalism of V-formulation is suitable for the classification of critical points and the derivation of imperfection sensitivity laws.

An imperfection parameter ξ is used for representing errors in nodal locations, member cross-sectional areas, and so on. The perfect system corresponds to $\xi = 0$. The critical load factor Λ^{c} is defined as the value of Λ at which an eigenvalue(s) of the stability matrix vanishes. We suppose the critical point has m-fold zero eigenvalues. The increment of Λ from Λ^{c} is denoted by $\widetilde{\Lambda}$.

The generalized coordinates $\mathbf{q} = (q_1, \dots, q_n)^\top$ in the direction of eigenmodes of the tangent stiffness matrix of the perfect system are decomposed into

- active coordinates $\mathbf{q}^{\mathrm{a}} = (q_1, \dots, q_m)^\top$ associated with zero eigenvalues, and
- passive coordinates $\mathbf{q}^{\mathrm{p}} = (q_{m+1}, \dots, q_n)^\top$ associated with nonzero ones,

namely, $\mathbf{q} = (\mathbf{q}^{\mathrm{a}\top}, \mathbf{q}^{\mathrm{p}\top})^\top$.

Then, by the implicit function theorem, $\mathbf{q}^{\mathrm{p}}$ can be locally expressed as a function of $\mathbf{q}^{\mathrm{a}}$, and the passive coordinates $\mathbf{q}^{\mathrm{p}}$ are eliminated from the total potential energy, which is expressed as the functions of $\mathbf{q}^{\mathrm{a}}$, $\widetilde{\Lambda}$ and ξ as $V(\mathbf{q}^{\mathrm{a}}, \widetilde{\Lambda}, \xi)$ [9, 28]. Hereafter, $q^{\mathrm{a}}_i = q_i$ $(i = 1, \dots, m)$ is used for simplicity.

The partial differentiation of V with respect to q^{a}_i, $\widetilde{\Lambda}$ and ξ are denoted by subscripts as $V_{,i}$, $V_{,\Lambda}$ and $V_{,\xi}$, respectively. Then $\Pi(\mathbf{q}^{\mathrm{a}}, \widetilde{\Lambda}, \xi)$ is expanded as

$$
\begin{aligned}
V(\mathbf{q}^{\mathrm{a}}, \widetilde{\Lambda}, \xi) \; = \; & V(\mathbf{0}, 0, 0) + \sum_{i=1}^{m} V_{,i} q_i + V_{,\Lambda} \widetilde{\Lambda} + V_{,\xi} \xi \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} V_{,ij} q_i q_j + \sum_{i=1}^{m} (V_{,i\Lambda} q_i \widetilde{\Lambda} + V_{,i\xi} q_i \xi) \\
& + \frac{1}{6} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} V_{,ijk} q_i q_j q_k + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} (V_{,ij\Lambda} q_i q_j \widetilde{\Lambda} + V_{,ij\xi} q_i q_j \xi) \\
& + \frac{1}{24} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} V_{,ijkl} q_i q_j q_k q_l + \text{h.o.t.}
\end{aligned}
\tag{4}
$$

where h.o.t. denotes higher-order terms. Note that all derivatives with respect to the generalized coordinates are evaluated with fixed direction of q_i at the critical point of the perfect system.

The stationary condition of V with respect to q_i, which is called *bifurcation equation*, is expressed as

$$
\frac{\partial V}{\partial q_i} = V_{,i} = 0, \quad (i = 1, \dots, m) \tag{5}
$$

3 Hilltop branching point

A hilltop branching point has one or more bifurcation point(s) at a limit point. We consider the case where a simple bifurcation point and a limit point coincide; i.e. $m = 2$, and assume q_1 and q_2 correspond to the bifurcation mode and limit-point mode, respectively. Hence, the derivatives of the potential V at this hilltop point satisfy the conditions

$$
V_{,1} = V_{,2} = 0 \tag{6a}
$$

$$
V_{,11} = V_{,12} = V_{,21} = V_{,22} = 0 \tag{6b}
$$

By the classification of the critical points, we have

$$
V_{,1\Lambda} = 0, \quad V_{,2\Lambda} \neq 0, \quad V_{,222} \neq 0 \tag{7}
$$

The imperfection is classified to *major imperfection* and *minor imperfection*, which are also called first-order imperfection and second-order imperfection, respectively [29]. A major imperfection has a first-order effect on the critical load of the imperfect systems, and the imperfection sensitivity of a bifurcation load is often unbounded for a major imperfection. On the contrary, a minor imperfection has a second-order effect, and the imperfection sensitivity of a bifurcation load is usually expressed as a linear function of an imperfection parameter [11] except for a special case of degenerate bifurcation point [16] including a degenerate hilltop point investigated in this paper.

For a hilltop point, minor and major imperfections are characterized by

- minor (symmetric) imperfection:

$$\begin{aligned} &V_{,1\xi} = 0, \quad V_{,12\xi} = 0, \ \ldots \\ &V_{,11\xi}, \quad V_{,2\xi}, \quad V_{,22\xi}, \ \cdots : \text{possibly nonzero} \end{aligned} \tag{8}$$

- major (antisymmetric) imperfection:

$$\begin{aligned} &V_{,11\xi} = 0, \quad V_{,2\xi} = 0, \quad V_{,22\xi} = 0 \\ &V_{,1\xi}, \quad V_{,12\xi}, \ \cdots : \text{possibly nonzero} \end{aligned} \tag{9}$$

which means that $V_{,\xi}$ is an even function of q_1 for the minor imperfection, and is an odd function of q_1 for the major imperfection; hence, the major imperfection has stronger effect on the total potential energy than the minor imperfection.

Assume that the perfect system has the trivial solution $q_1 = 0$, namely, the first-order terms of V with respect to q_1 vanish at $\xi = 0$ as follows:

$$V_{,122} = V_{,12\Lambda} = V_{,1\Lambda\Lambda} = \cdots = 0 \tag{10}$$

In the following we assume

$$V_{,11\Lambda} \neq 0, \quad V_{,1122} \neq 0 \tag{11}$$

The total potential energy is expanded at the hilltop point of the perfect system $(\mathbf{q}, \widetilde{\Lambda}, \xi) = (\mathbf{0}, 0, 0)$ as

$$\begin{aligned} V(\mathbf{q}^{\mathrm{a}}, \widetilde{\Lambda}, \xi) = &V(\mathbf{0}, 0, 0) + V_{,2\Lambda} q_2 \widetilde{\Lambda} + V_{,1\xi} q_1 \xi + V_{,2\xi} q_2 \xi \\ &+ \frac{1}{6} V_{,111} (q_1)^3 + \frac{1}{2} V_{,112} (q_1)^2 q_2 + \frac{1}{6} V_{,222} (q_2)^3 \\ &+ \frac{1}{2} V_{,11\Lambda} (q_1)^2 \widetilde{\Lambda} + \frac{1}{2} V_{,22\Lambda} (q_2)^2 \widetilde{\Lambda} \\ &+ \frac{1}{2} V_{,11\xi} (q_1)^2 \xi + V_{,12\xi} q_1 q_2 \xi + \frac{1}{2} V_{,22\xi} (q_2)^2 \xi \\ &+ \frac{1}{24} V_{,1111} (q_1)^4 + \frac{1}{4} V_{,1122} (q_1)^2 (q_2)^2 \\ &+ \frac{1}{6} V_{,1112} (q_1)^3 q_2 + \frac{1}{2} V_{,112\Lambda} (q_1)^2 q_2 \widetilde{\Lambda} + \text{h.o.t.} \end{aligned} \tag{12}$$

where the term $V_{,2222}$ is suppressed in view of the presence of $V_{,222}$.

4 Perfect behavior of degenerate hilltop point.

The degenerate hilltop point is defined and the perfect behavior ($\xi = 0$) in the neighborhood of this point is investigated. Higher-order terms, h.o.t., are often suppressed in the sequel.

4.1 General formulations

The set of bifurcation equations is obtained as

$$\begin{aligned}\frac{\partial V}{\partial q_1} &= \frac{1}{2}V_{,111}(q_1)^2 + V_{,112}q_1q_2 + V_{,11\Lambda}q_1\widetilde{\Lambda} + \frac{1}{6}V_{,1111}(q_1)^3 \\ &\quad + \frac{1}{2}V_{,1122}q_1(q_2)^2 + \frac{1}{2}V_{,1112}(q_1)^2q_2 + \text{h.o.t.} = 0\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial V}{\partial q_2} &= V_{,2\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,112}(q_1)^2 + \frac{1}{2}V_{,222}(q_2)^2 \\ &\quad + \frac{1}{2}V_{,1122}(q_1)^2q_2 + \frac{1}{6}V_{,1112}(q_1)^3 + \text{h.o.t.} = 0\end{aligned} \tag{14}$$

Although the terms $V_{,1111}$, $V_{,1122}$ and $V_{,1112}$ seem to be redundant, they are considered in view of possibility that $V_{,111}$ and $V_{,112}$ may vanish. The criticality condition in (2) is given with the expression of the stability matrix of the bifurcation equation

$$\mathbf{S}(q_1, q_2, \widetilde{\Lambda}) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{15}$$

with

$$S_{11} = V_{,111}q_1 + V_{,112}q_2 + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2 + V_{,1112}q_1q_2 + \text{h.o.t.} \tag{16a}$$

$$S_{12} = S_{21} = V_{,112}q_1 + V_{,1122}q_1q_2 + \frac{1}{2}V_{,1112}(q_1)^2 + V_{,112\Lambda}q_1\widetilde{\Lambda} + \text{h.o.t.} \tag{16b}$$

$$S_{22} = V_{,222}q_2 + V_{,22\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1122}(q_1)^2 + \text{h.o.t.} \tag{16c}$$

Note that the coefficients in (16) should be found by twice differentiating V so that higher-order terms suppressed in (13) and (14) turn out to be the leading-order terms in (16). Hereafter we carry out leading-order asymptotic analyses and "h.o.t." is omitted for simplicity.

4.2 Degenerate hilltop point

Consider the case where an asymmetric bifurcation point exits at the limit point; i.e. $V_{,111} \neq 0$. Eq. (13) is factorized as

$$q_1\Big(\frac{1}{2}V_{,111}q_1 + V_{,112}q_2 + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1122}(q_2)^2\Big) = 0 \tag{17}$$

Therefore, we have the following two solutions:

- trivial fundamental path:

$$q_1 = 0 \tag{18}$$

- bifurcation path:

$$\frac{1}{2}V_{,111}q_1 + V_{,112}q_2 + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \tag{19}$$

Note that the bifurcation path is always existent.

In the following, the path that contains the undeformed initial state is called fundamental path; the path bifurcates from the fundamental path is called bifurcation path; and the path that cannot be reached from the initial state is called an aloof path.

From (14) and (18), we have the fundamental path parameterized by q_2 as

$$\widetilde{\Lambda} = -\frac{V_{,222}}{2V_{,2\Lambda}}(q_2)^2 \tag{20}$$

From (16), the stability matrix $\mathbf{S}$ in (15) on this trivial path with $q_1 = 0$ reduces to

$$\mathbf{S}(0, q_2, \widetilde{\Lambda}) = \begin{pmatrix} \lambda_\alpha & 0 \\ 0 & \lambda_\beta \end{pmatrix} \tag{21}$$

with two eigenvalues (cf., (20))

$$\lambda_\alpha = V_{,112}q_2 + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1122}(q_2)^2 = V_{,112}q_2 + C_\alpha(q_2)^2 \tag{22a}$$

$$\lambda_\beta = V_{,222}q_2 + V_{,22\Lambda}\widetilde{\Lambda} = V_{,222}q_2 - \frac{V_{,222}V_{,122\Lambda}}{2V_{,2\Lambda}}(q_2)^2 \tag{22b}$$

Here C_α is defined as

$$C_\alpha = \frac{1}{2V_{,2\Lambda}}(V_{,2\Lambda}V_{,1122} - V_{,222}V_{,11\Lambda}) \tag{23}$$

and assume

$$C_\alpha > 0 \implies V_{,1122} > \frac{V_{,222}V_{,11\Lambda}}{V_{,2\Lambda}} \tag{24}$$

The eigenvalue λ_α in (22a) becomes degenerate for $V_{,112} = 0$ in the sense that λ_α is tangential to the q_2-axis at $q_2 = 0$. Hence, the degeneracy of a hilltop point is characterized by

$$V_{,112} = 0, \quad V_{,1122} \neq 0 \tag{25}$$

From (7), λ_β in (22b) simplifies to

$$\lambda_\beta = V_{,222}q_2 \tag{26}$$

Consider the most customary case in practice where the system has a rising fundamental path and becomes unstable at the hilltop point. Then from (20) and (26), we have

$$V_{,222} < 0, \quad V_{,2\Lambda} < 0 \tag{27}$$

We encounter complex behavior in the vicinity of this degenerate hilltop point due to vanishing of many differential coefficients. The two eigenvalues λ_α and λ_β in (22a) and (26), respectively, behave as shown in Fig. 1. The curve of λ_α is tangential to the q_2-axis, and λ_α is positive except for the hilltop point. In the presence of a small imperfection, λ_α may cease to vanish at the degenerate hilltop point to trigger catastrophic change that entails difficulties in imperfection sensitivity analysis, including a discontinuity in the

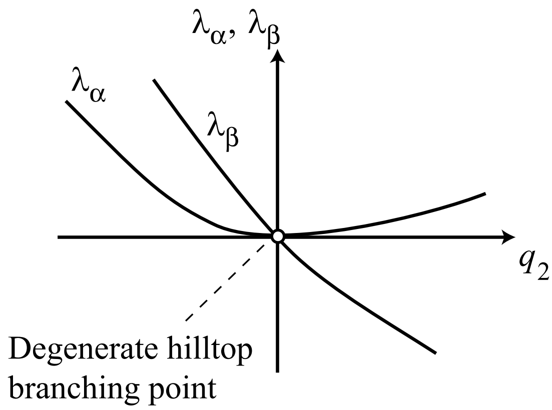

Fig. 1. Variation of the two lowest eigenvalues λ_α and λ_β that vanish at a degenerate hilltop branching point ○.

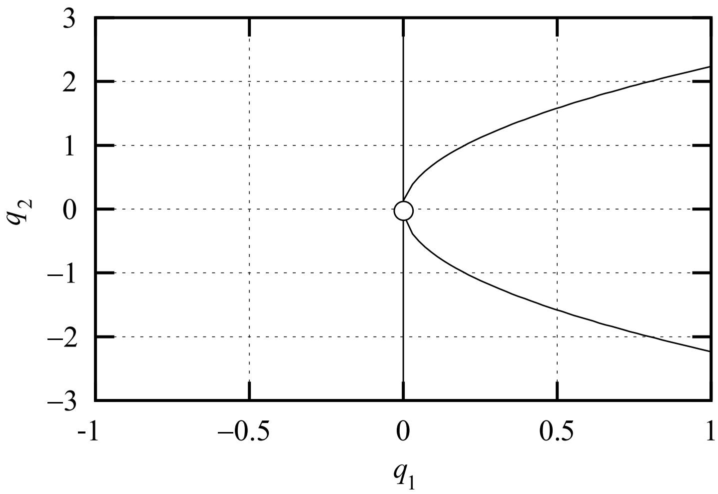

Fig. 2. Asymptotic behavior of the perfect system in the vicinity of a degenerate hilltop branching point with asymmetric bifurcation. ○: hilltop point; $V_{,2\Lambda} = -1.0$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,1111} = -0.01$, $V_{,1122} = 0.01$.

critical point and its unbounded sensitivity coefficient with respect to the imperfection parameter [16].

From (14) and (19) with $V_{,112} = 0$, we have the bifurcation path parameterized by q_2 as

$$\widetilde{\Lambda} = -\frac{V_{,222}}{2V_{,2\Lambda}}(q_2)^2 \leq 0 \tag{28a}$$

$$q_1 = -\frac{2C_\alpha}{V_{,111}}(q_2)^2 \tag{28b}$$

Note that (28a) is same as (20) for the fundamental path. The perfect behavior of a degenerate hilltop point with asymmetric bifurcation is illustrated in Fig. 2.

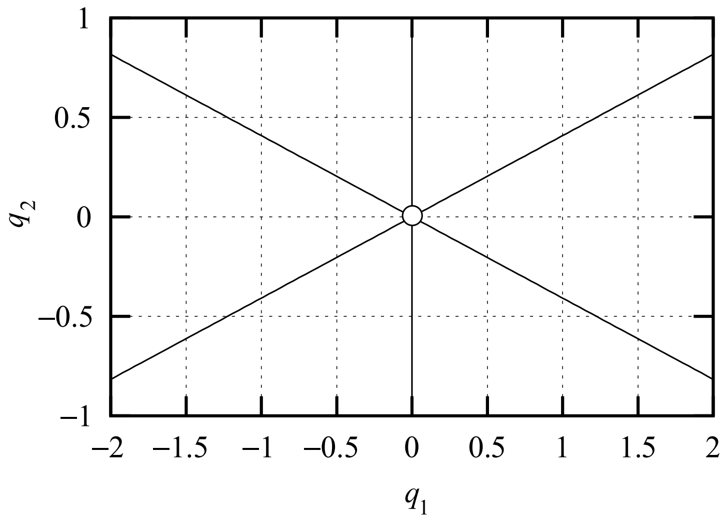

Fig. 3. Asymptotic behaviors of the perfect system in the vicinity of a degenerate hilltop branching point with symmetric bifurcation. ○: hilltop point; $V_{,2\Lambda} = -1.0$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,111} = -0.1$, $V_{,1122} = 0.01$.

For a hilltop point with symmetric bifurcation with $V_{,111} = 0$ and $V_{,1111} \neq 0$, the two bifurcation paths are parameterized by q_2 as

$$\widetilde{\Lambda} = -\frac{V_{,222}}{2V_{,2\Lambda}}(q_2)^2 \leq 0, \tag{29a}$$

$$q_1 = \pm\Big(-\frac{6C_\alpha}{V_{,1111}}\Big)^{\frac{1}{2}} q_2 \tag{29b}$$

that are conditionally existent for $V_{,1111} < 0$, i.e., for a declining bifurcation path.

The perfect behavior of a degenerate hilltop point with unstable-symmetric bifurcation is illustrated in Fig. 3.

5 Imperfection Sensitivity Laws I: Asymmetric Bifurcation

The imperfection sensitivity laws of degenerate hilltop points with simple asymmetric bifurcation are derived. It will turn out that $S_{11} = 0$ corresponds to a bifurcation point and S_{22} to a limit point on the fundamental path. The consistent set of the orders of the variables at the critical points of imperfect systems is found by the vertex enumeration of the feasible region of the solutions, where Maple 11 [30] has been used for symbolic computation. See Appendix 1 for detail.

5.1 General formulation

The bifurcation equations are obtained as

$$\frac{\partial V}{\partial q_1} = V_{,1\xi}\xi + \frac{1}{2}V_{,111}(q_1)^2 + V_{,11\Lambda}q_1\widetilde{\Lambda} + V_{,11\xi}q_1\xi + V_{,12\xi}q_2\xi + \frac{1}{2}V_{,1122}q_1(q_2)^2 = 0 \tag{30}$$

$$\frac{\partial V}{\partial q_2} = V_{,2\xi}\xi + V_{,2\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,222}(q_2)^2 + V_{,12\xi}q_1\xi + \frac{1}{2}V_{,1122}(q_1)^2 q_2 = 0 \tag{31}$$

The components of the stability matrix are given as

$$S_{11} = V_{,111}q_1 + V_{,11\Lambda}\widetilde{\Lambda} + V_{,11\xi}\xi + \frac{1}{2}V_{,1122}(q_2)^2 \tag{32a}$$

$$S_{12} = S_{21} = V_{,12\xi}\xi + V_{,1122}q_1 q_2 + \frac{1}{2}V_{,1112}(q_1)^2 \tag{32b}$$

$$S_{22} = V_{,222}q_2 + V_{,22\Lambda}\widetilde{\Lambda} + V_{,22\xi}\xi + \frac{1}{2}V_{,1122}(q_1)^2 \tag{32c}$$

5.2 Imperfection sensitivity: minor symmetric

Consider a symmetric minor imperfection and assume

$$V_{,1\xi} = 0, \quad V_{,12\xi} = 0, \quad V_{,111} < 0, \quad V_{,11\xi} < 0 \tag{33}$$

Eq. (30) has the following two solutions:

- trivial fundamental path

$$q_1 = 0 \tag{34}$$

- bifurcation or aloof path

$$\frac{1}{2}V_{,111}q_1 + V_{,11\Lambda}\widetilde{\Lambda} + V_{,11\xi}\xi + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \tag{35}$$

For the trivial fundamental path $q_1 = 0$, (31) reduces to

$$\widetilde{\Lambda} = -\frac{1}{V_{,2\Lambda}}\Big(V_{,2\xi}\xi + \frac{1}{2}V_{,222}(q_2)^2\Big) \tag{36}$$

Furthermore, from $V_{,12\xi} = 0$ and $q_1 = 0$, S_{12} in (32b) vanishes and $S_{11} = 0$ or $S_{22} = 0$ should hold at a critical point on the fundamental path. From $S_{22} = 0$, we obtain the location of a limit point

$$\widetilde{\Lambda}^{\rm c}_{\rm Lim}(\xi) = -\frac{V_{,2\xi}}{V_{,2\Lambda}}\xi, \quad q_1^{\rm c} = 0, \quad q_2^{\rm c} = 0 \tag{37}$$

where $(\,\cdot\,)^{\rm c}$ denotes a value at a critical point.

From $S_{11} = 0$ and (36), we obtain the location of a bifurcation point as

$$\widetilde{\Lambda}^{\rm c}_{\rm Bif}(\xi) = -\frac{V_{,2\xi}V_{,1122} - V_{,222}V_{,11\xi}}{2V_{,2\Lambda}C_\alpha}\xi \tag{38}$$

that is existent for ξ that satisfies

$$(q_2)^2 = \beta\xi > 0 \tag{39}$$

with

$$\beta = \frac{V_{,2\xi}V_{,11\Lambda} - V_{,2\Lambda}V_{,11\xi}}{2V_{,2\Lambda}C_\alpha} \tag{40}$$

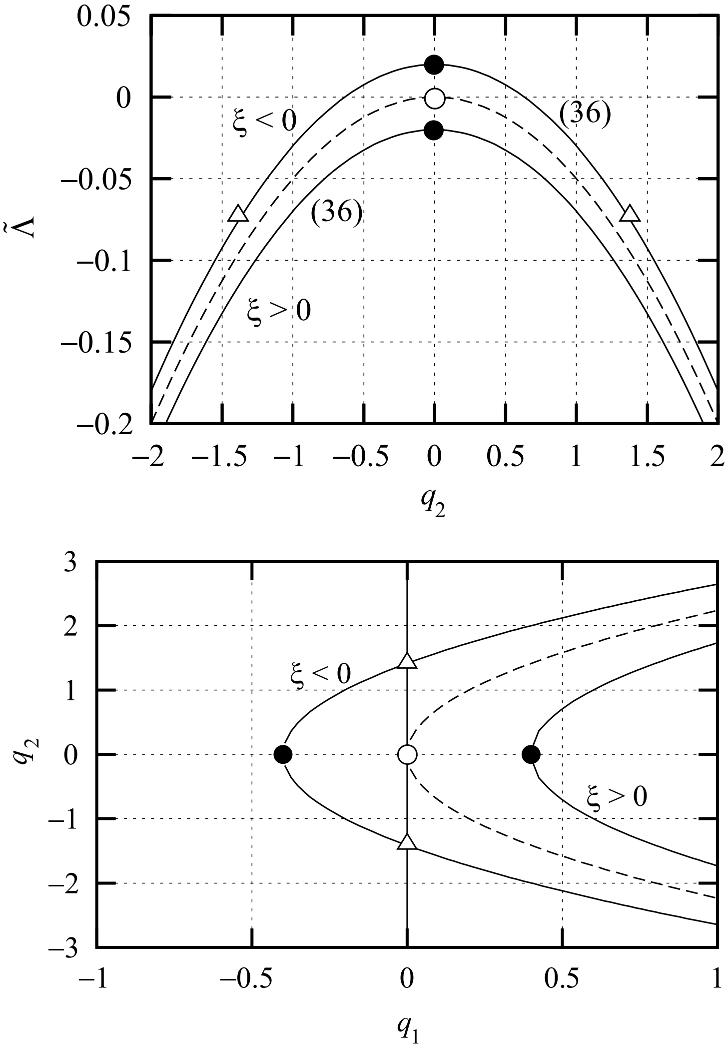

Fig. 4. Asymptotic behaviors of the imperfect system with a minor imperfection in the vicinity of a degenerate hilltop branching point with asymmetric bifurcation. ○: hilltop point; ●: limit point; △: bifurcation point; dashed curve: perfect equilibrium path; solid curve: imperfect equilibrium path. $\xi = \pm 0.1$, $V_{,2\xi} = -1.0$, $V_{,22\xi} = -0.1$, $V_{,11\xi} = 0.1$, $V_{,2\Lambda} = -1.0$, $V_{,22\Lambda} = -0.1$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,111} = -0.1$, $V_{,1122} = 0.01$.

For the bifurcation or aloof path (35), the set of equations (31) and (35) yields the $q_1 - q_2$ curve as

$$\Big(\frac{V_{,11\xi}V_{,2\Lambda} - V_{,11\Lambda}V_{,2\xi}}{V_{,2\Lambda}}\Big)\xi + \frac{1}{2}V_{,111}q_1 + C_\alpha(q_2)^2 = 0 \tag{41}$$

In an asymptotic sense, the two terms $V_{,12\xi}q_1\xi$ and $V_{,1122}(q_1)^2q_2/2$ in (31) vanishes as higher order terms, and (31) reduces to (36); therefore, these paths have the same $\widetilde{\Lambda} - q_2$ curve as the fundamental path.

It can be easily confirmed that the bifurcation path intersects with the fundamental path at the bifurcation points defined by (38) and (39) for $\beta\xi > 0$.

By the vertex enumeration of the feasible region of solutions, we obtain the two feasible set of orders to the set of equations (30), (31) and $\det \mathbf{S} = 0$ with (32). These two sets are called vertices 1 and 4 in Table A.1. The vertex 1 corresponds to the limit point on the fundamental path in (37). From the vertex 4, we obtain the location of a limit point on the bifurcation or aloof path as

$$q_1^{\mathrm{c}}(\xi) = \frac{2(V_{,11\Lambda}V_{,2\xi} - V_{,2\Lambda}V_{,11\xi})}{V_{,111}V_{,2\Lambda}}\xi \tag{42a}$$

$$q_2^{\mathrm{c}}(\xi) = \frac{V_{,22\Lambda}V_{,2\xi} - V_{,2\Lambda}V_{,22\xi}}{V_{,222}V_{,2\Lambda}}\xi \tag{42b}$$

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = -\frac{V_{,2\xi}}{V_{,2\Lambda}}\xi \tag{42c}$$

The equilibrium paths and the critical points are shown in Fig. 4 with the associated equation number.

5.3 Imperfection sensitivity: major antisymmetric

Consider an antisymmetric major imperfection and assume

$$V_{,11\xi} = 0, \quad V_{,2\xi} = 0, \quad V_{,22\xi} = 0, \quad V_{,111} < 0, \quad V_{,1\xi} < 0 \tag{43}$$

From (30), (31) and (43), we obtain the $q_1 - q_2$ curve as

$$V_{,1\xi}\xi + \frac{1}{2}V_{,111}(q_1)^2 + C_\alpha q_1 (q_2)^2 = 0 \tag{44}$$

in which higher-order terms are suppressed, and $\widetilde{\Lambda}$ is expressed with respect to q_2 as

$$\widetilde{\Lambda} = -\frac{V_{,222}}{2V_{,2\Lambda}}(q_2)^2 + \frac{V_{,1122}V_{,1\xi}}{N_{,2\Lambda}V_{,111}}q_2\xi \tag{45}$$

As shown in Table A.2 in Appendix 1, the set of equations (30), (31) and $\det \mathbf{S} = 0$ with (32) has two solutions of different orders

$$\text{I)} \quad \widetilde{\Lambda} = O(\xi^{\frac{1}{2}}), \quad q_1 = O(\xi^{\frac{1}{2}}), \quad q_2 = O(\xi^{\frac{1}{4}}) \tag{46}$$

$$\text{II)} \quad \widetilde{\Lambda} = O(\xi^{\frac{3}{2}}), \quad q_1 = O(\xi^{\frac{1}{2}}), \quad q_2 = O(\xi) \tag{47}$$

5.3.1 Type I solution

For Type I solution in (46), the set of equations (30), (31) and $\det \mathbf{S} = 0$ with (32) reduces to (A.13)–(A.15) in Appendix 2, where the higher-order terms have been suppressed and $S_{12} = 0$ holds.

The solutions of the set of equations (A.13)–(A.15) give the locations of two limit points of an imperfect system as

$$q_1^{\mathrm{c}}(\xi) = -\operatorname{sgn}(V_{,111})\Big(\frac{2V_{,1\xi}\xi}{V_{,111}}\Big)^{\frac{1}{2}} \tag{48a}$$

$$q_2^{\mathrm{c}}(\xi) = \pm\frac{(2V_{,111}V_{,1\xi}\xi)^{\frac{1}{4}}}{(C_\alpha)^{\frac{1}{2}}} \tag{48b}$$

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = -\frac{V_{,222}}{2V_{,2\Lambda}C_\alpha}(2V_{,111}V_{,1\xi}\xi)^{\frac{1}{2}} < 0 \tag{48c}$$

where $\operatorname{sgn}(V_{,111})$ is the sign of $V_{,111}$, and the solutions (48a)–(48c) exist for

$$V_{,111}V_{,1\xi}\xi > 0 \tag{49}$$

5.3.2 Type II solution

For Type II solution in (47), the set of equations (30), (31) and $\det \mathbf{S} = 0$ with (32) reduces to (A.16)–(A.18) in Appendix 2.

The solutions of the set of equations (A.16)–(A.18) give the locations of two limit points of an imperfect system as

$$q_1^{\mathrm{c}}(\xi) = \pm\Big(-\frac{2V_{,1\xi}\xi}{V_{,111}}\Big)^{\frac{1}{2}} \tag{50a}$$

$$q_2^{\mathrm{c}}(\xi) = \frac{V_{,1122}V_{,1\xi}}{V_{,111}V_{,222}}\xi \tag{50b}$$

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = \mp\frac{\operatorname{sgn}(\xi)V_{,12\xi}}{V_{,2\Lambda}}\Big(-\frac{2\operatorname{sgn}(\xi)V_{,1\xi}}{V_{,111}}\Big)^{\frac{1}{2}}|\xi|^{\frac{3}{2}} \tag{50c}$$

that are existent for

$$V_{,111}V_{,1\xi}\xi < 0 \tag{51}$$

Here the double signs take the same order. One of these solutions corresponds to the limit point on an imperfect fundamental path, and the other to that on an aloof path.

The imperfect equilibrium paths and the critical points are shown in Fig. 5.

6 Imperfection Sensitivity Laws II: Unstable-Symmetric Bifurcation

The imperfection sensitivity laws of degenerate hilltop points with simple unstable-symmetric bifurcation are derived.

6.1 General formulation

We consider a degenerate hilltop point with simple unstable-symmetric bifurcation with

$$V_{,111} = V_{,112} = 0, \quad V_{,1122} \neq 0 \tag{52}$$

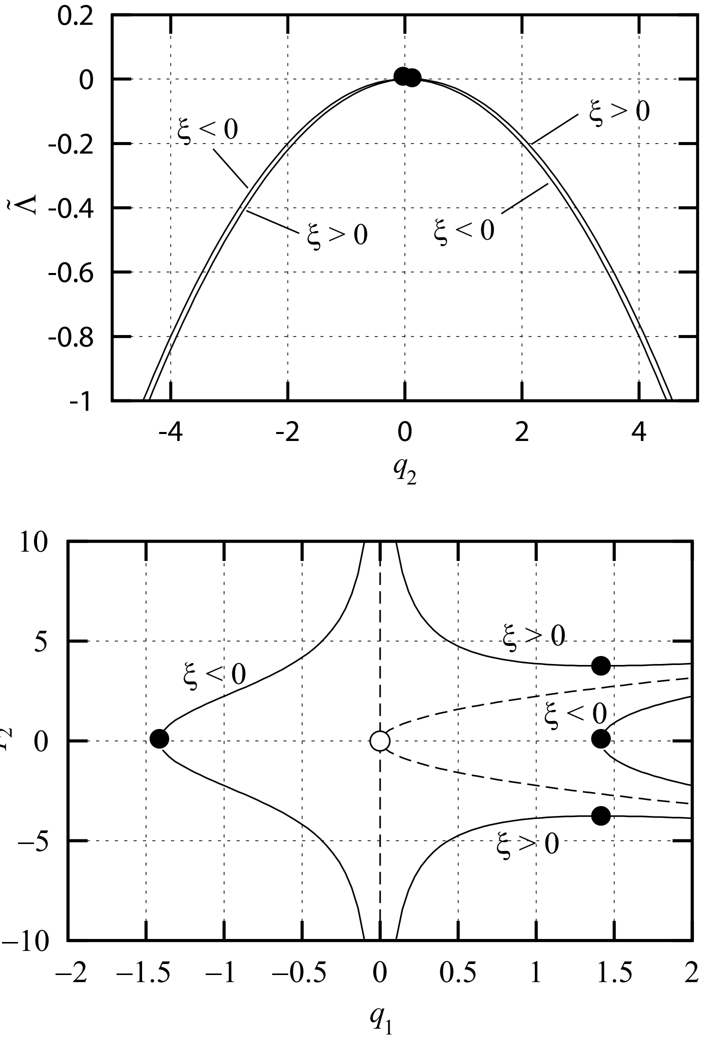

Fig. 5. Asymptotic behaviors of the imperfect system with a major imperfection in the vicinity of a degenerate hilltop branching point with asymmetric bifurcation. ○: hilltop point; •: limit point; dashed curve: perfect equilibrium path; solid curve: imperfect equilibrium path. $\xi = \pm 0.01$, $V_{,1\xi} = -1.0$, $V_{,12\xi} = -0.1$, $V_{,2\Lambda} = -1.0$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,111} = -0.1$, $V_{,1122} = 0.01$.

and focus on the most customary case in practice where the system has a rising fundamental path and becomes unstable at the hilltop point. Then we have (cf., (23) and (27))

$$V_{,2\Lambda} < 0, \quad V_{,222} < 0, \quad V_{,1111} < 0, \quad V_{,22\Lambda} < 0 \tag{53}$$

$$C_\alpha = \frac{1}{2V_{,2\Lambda}}(V_{,2\Lambda}V_{,1122} - V_{,222}V_{,11\Lambda}) > 0 \tag{54}$$

With the use of (52), the bifurcation equations become

$$\frac{\partial V}{\partial q_1} = V_{,1\xi}\xi + V_{,11\Lambda}q_1\widetilde{\Lambda} + V_{,11\xi}q_1\xi + V_{,12\xi}q_2\xi + \frac{1}{6}V_{,1111}(q_1)^3 + \frac{1}{2}V_{,1122}q_1(q_2)^2 = 0 \quad (55)$$

$$\frac{\partial V}{\partial q_2} = V_{,2\Lambda}\widetilde{\Lambda} + V_{,2\xi}\xi + \frac{1}{2}V_{,222}(q_2)^2 + V_{,22\Lambda}q_2\widetilde{\Lambda} + V_{,12\xi}q_1\xi + \frac{1}{2}V_{,1122}(q_1)^2 q_2 = 0 \quad (56)$$

and the components of the stability matrix are written as

$$S_{11} = V_{,11\xi}\xi + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2 \quad (57a)$$

$$S_{12} = S_{21} = V_{,12\xi}\xi + V_{,1122}q_1 q_2 \quad (57b)$$

$$S_{22} = V_{,222}q_2 + V_{,22\Lambda}\widetilde{\Lambda} + V_{,22\xi}\xi + \frac{1}{2}V_{,1122}(q_1)^2 \quad (57c)$$

6.2 Imperfection sensitivity: minor symmetric

Consider a symmetric minor imperfection and assume

$$V_{,1\xi} = 0, \quad V_{,12\xi} = 0, \quad V_{,111} < 0, \quad V_{,11\xi} < 0 \quad (58)$$

From (55) and (56), we obtain the equilibrium path as

$$\frac{V_{,11\xi}V_{,2\Lambda} - V_{,11\Lambda}V_{,2\xi}}{V_{,2\Lambda}}\xi + \frac{1}{6}V_{,1111}(q_1)^2 + C_\alpha(q_2)^2 = 0 \quad (59)$$

where higher-order terms have been removed.

Eq. (55) has two solutions:

- trivial fundamental path:

$$q_1 = 0 \quad (60)$$

- bifurcation or aloof path:

$$V_{,11\Lambda}\widetilde{\Lambda} + V_{,11\xi}\xi + \frac{1}{6}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \quad (61)$$

For the trivial fundamental path $q_1 = 0$, (56) leads to

$$\widetilde{\Lambda} = -\frac{1}{V_{,2\Lambda}}\left(V_{,2\xi}\xi + \frac{1}{2}V_{,222}(q_2)^2\right) \quad (62)$$

The limit point load is given for $q_2 = 0$ as

$$\widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Lim}}(\xi) = -\frac{V_{,2\xi}}{V_{,2\Lambda}}\xi \quad (63)$$

By the vertex enumeration of the feasible region of solutions, we obtain the feasible set of solutions to the set of equations (55), (56) and $\det \mathbf{S} = 0$ with (57).

Eq. (63) splits into two solutions $q_2 = 0$ and $q_2 \neq 0$. The vertex 3 in Table A.3 corresponds to the limit point on the fundamental path. From the vertex 1 in Table A.3, Eqs. (55)–(57) are reduced to the following equations (A.19)–(A.21) in Appendix 2 for the bifurcation or aloof path (61).

The bifurcation point is obtained from (A.19)–(A.21) with $q_2 \neq 0$ as

$$\widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Bif}}(\xi) = -\frac{V_{,2\xi}V_{,1122} - V_{,222}V_{,11\xi}}{2V_{,2\Lambda}C_\alpha}\xi \tag{64}$$

that is existent for

$$(q_2)^2 = \beta\xi > 0 \tag{65}$$

otherwise no bifurcation point exists.

To sum up,

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = \begin{cases} \widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Lim}}(\xi): & \text{for } \beta\xi < 0 \\ \widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Bif}}(\xi): & \text{for } \beta\xi > 0 \end{cases} \tag{66}$$

The limit point on the aloof path is obtained from (A.19)–(A.21) with $q_2 = 0$ as

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = -\frac{V_{,2\xi}}{V_{,2\Lambda}}\xi \tag{67}$$

which exists for

$$(q_1)^2 = \frac{6(V_{,11\Lambda}V_{,2\xi} - V_{,2\Lambda}V_{,11\xi})}{V_{,2\Lambda}V_{,1111}}\xi > 0 \tag{68}$$

Note that

$$\widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Lim}} - \widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Bif}} = \frac{V_{,222}}{2V_{,2\Lambda}}\beta\xi > 0 \tag{69}$$

The imperfect equilibrium paths and the critical points are shown in Fig. 6, where the equation numbers are also indicated.

6.3 Imperfection sensitivity: major antisymmetric

Consider an antisymmetric major imperfection and assume

$$V_{,11\xi} = 0, \quad V_{,2\xi} = 0, \quad V_{,22\xi} = 0, \quad V_{,1111} < 0, \quad V_{,1\xi} < 0 \tag{70}$$

From (55), (56) and (70), we obtain the following equilibrium path:

$$V_{,1\xi}\xi + \frac{1}{6}V_{,1111}(q_1)^3 + C_\alpha q_1 (q_2)^2 = 0 \tag{71}$$

where the higher-order terms have been removed.

By the vertex enumeration of the feasible region of solutions, we obtain the feasible set of solutions to the set of equations (55), (56) and $\det \mathbf{S} = 0$ with (57). From the vertices 2 and 7 in Table A.4, we have two solutions of different orders as

$$\text{I)} \quad \widetilde{\Lambda} = O(\xi^{\frac{2}{3}}), \quad q_1 = O(\xi^{\frac{1}{3}}), \quad q_2 = O(\xi^{\frac{1}{3}}) \tag{72}$$

$$\text{II)} \quad \widetilde{\Lambda} = O(\xi^{\frac{4}{3}}), \quad q_1 = O(\xi^{\frac{1}{3}}), \quad q_2 = O(\xi^{\frac{2}{3}}) \tag{73}$$

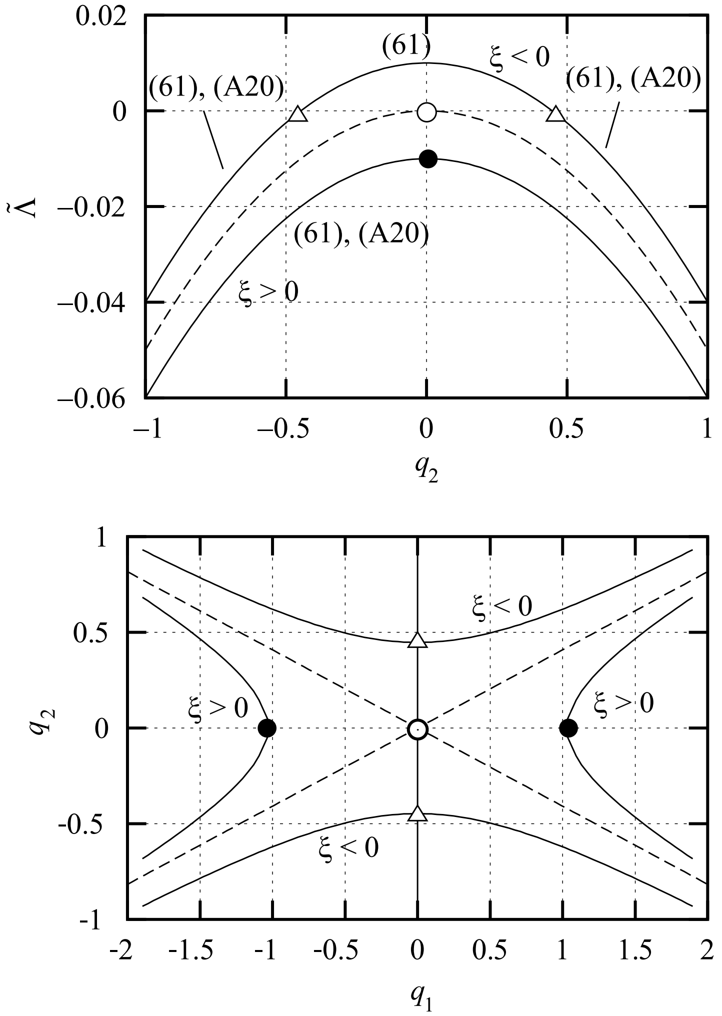

Fig. 6. Asymptotic behaviors of the imperfect system with a minor imperfection in the vicinity of a degenerate hilltop branching point with symmetric bifurcation. ○: hilltop point; •: limit point; △: unstable-symmetric bifurcation point; dashed curve: perfect equilibrium path; solid curve: imperfect equilibrium path. $\xi = \pm 0.01$, $V_{,2\xi} = -1.0$, $V_{,2\Lambda} = -1.0$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,1111} = -0.01$, $V_{,1122} = 0.01$.

6.3.1 Type I solution

For the Type I solution in (72), the bifurcation equations (55), (56) and the criticality condition reduce to (A.22)–(A.24) in Appendix 2.

The solutions of the set of equations (A.22)–(A.24) give the locations of two limit points of an imperfect system as

$$q_1^{\mathrm{c}}(\xi) = \left(\frac{3}{V_{,1111}}\right)^{\frac{1}{3}} (V_{,1\xi}\xi)^{\frac{1}{3}} \tag{74a}$$

$$(q_2^{\mathrm{c}}(\xi))^2 = -\frac{3^{\frac{2}{3}}(V_{,1111})^{\frac{1}{3}}}{2C_\alpha}(V_{,1\xi}\xi)^{\frac{2}{3}} \tag{74b}$$

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = \frac{3^{\frac{2}{3}} V_{,222}(V_{,1111})^{\frac{1}{3}}}{4V_{,2\Lambda}C_\alpha}(V_{,1\xi}\xi)^{\frac{2}{3}} < 0 \tag{74c}$$

One of these solutions corresponds to the limit point on an imperfect fundamental path, and the other to that on an aloof path.

6.3.2 Type II solution

For the Type II solution in (73), the bifurcation equations (55), (56) and the criticality condition reduce to (A.25)–(A.27) in Appendix 2.

The solution of the set of equations (A.25)–(A.27) gives the locations of a limit point of an imperfect system as

$$q_1^{\mathrm{c}}(\xi) = -\left(\frac{6V_{,1\xi}\xi}{V_{,1111}}\right)^{\frac{1}{3}} \tag{75a}$$

$$q_2^{\mathrm{c}}(\xi) = \frac{V_{,1122}}{2V_{,222}}\left(\frac{6V_{,1\xi}\xi}{V_{,1111}}\right)^{\frac{2}{3}} \tag{75b}$$

$$\widetilde{\Lambda}^{\mathrm{c}}(\xi) = \left[-\frac{(V_{,1122})^2}{8V_{,2\Lambda}V_{,222}}\left(\frac{6V_{,1\xi}}{V_{,1111}}\right)^{\frac{4}{3}} + \frac{1}{V_{,2\Lambda}}\left(\frac{6V_{,1\xi}}{V_{,1111}}\right)^{\frac{1}{3}}V_{,12\xi}\right]\xi^{\frac{4}{3}} \tag{75c}$$

This solution corresponds to the limit point on an aloof path.

The imperfect equilibrium paths and the critical points are shown in Fig. 7.

7 Numerical examples

Consider the four-bar truss tent as shown in Fig. 8 subjected to a proportional vertical load with $P = 1000.0$ N, where $L = 1000$ mm. In the following, the units of length and force are omitted for brevity. The four members are composed of linear elastic material with elastic modulus $E = 200.0$. The cross-sectional areas are 1000.0 for all members.

The length of each member at the deformed state is computed exactly from the deformed nodal locations to obtain the engineering strain as the elongation divided by the member length at the undeformed state. Therefore, the member strains are expressed as explicit functions of nodal displacements, and the differential coefficients of V are easily computed using the symbolic computation package Maple 11 [30]. See, e.g., [31] for details.

The force F and extension d of spring 1 have a nonlinear relation with parameters K_2 and K_3 as

$$F = K_2 d^2 + K_3 d^3 \tag{76}$$

Spring 2 has linear extensional stiffness 10.0. As demonstrated in Ref. [10], the truss has a degenerate hilltop branching point if $H = 1541.10$, where the critical load factor is 153.96.

Let $\boldsymbol{\Phi} = (\Phi_x, \Phi_y, \Phi_z)^\top$ denote the buckling mode, where Φ_x, Φ_y and Φ_z are the displacements in x-, y- and z-directions, respectively, of the top node. The limit point mode $\boldsymbol{\Phi}^{\mathrm{L}}$ and the bifurcation mode $\boldsymbol{\Phi}^{\mathrm{B}}$ are obtained as

$$\boldsymbol{\Phi}^{\mathrm{L}} = (0, 0, 1)^\top, \quad \boldsymbol{\Phi}^{\mathrm{B}} = (1, 0, 0)^\top \tag{77}$$

The imperfection in the location of the top node is considered. The variation of nodal coordinates in the directions of $\boldsymbol{\Phi}^{\mathrm{L}}$ and $\boldsymbol{\Phi}^{\mathrm{B}}$ correspond to minor and major imperfections, respectively.

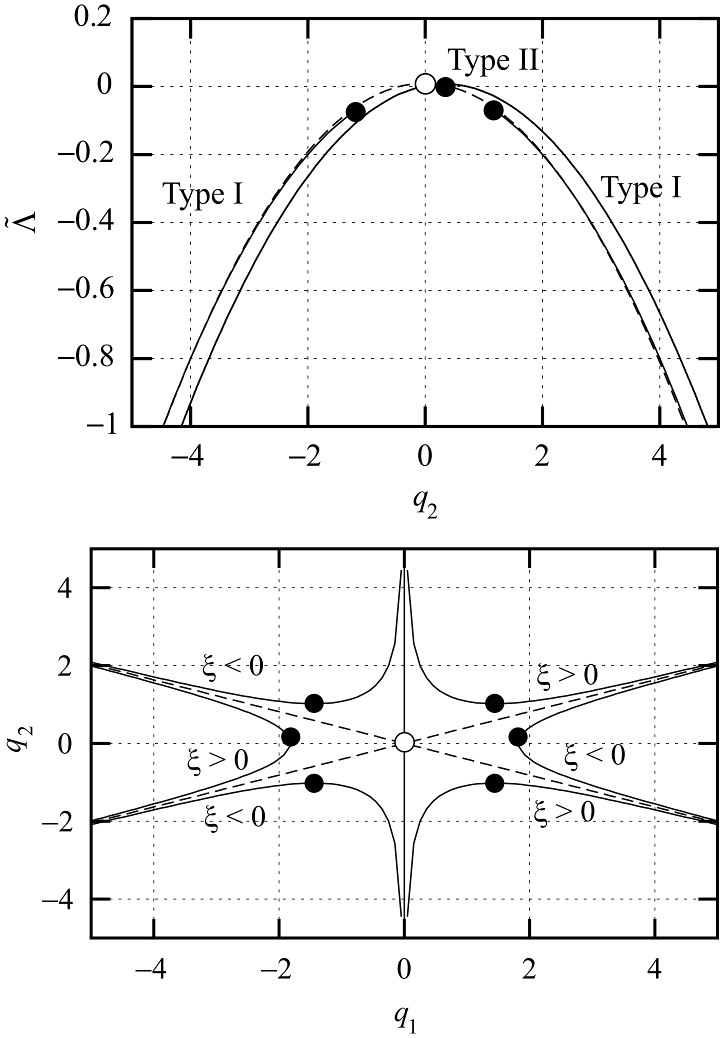

Fig. 7. Asymptotic behaviors of the imperfect system with a major imperfection in the vicinity of a degenerate hilltop branching point with symmetric bifurcation. ○: hilltop point; •: limit point; dashed curve: perfect equilibrium path; solid curve: imperfect equilibrium path. $\xi = \pm 0.1$, $V_{,1\xi} = -1.0$, $V_{,2\Lambda} = -1.0$, $V_{,22\Lambda} = -0.1$, $V_{,222} = -0.1$, $V_{,11\Lambda} = -0.1$, $V_{,111} = -0.1$, $V_{,1122} = 0.01$.

In the following, imperfection sensitivity of the first critical point along the fundamental path is computed by path-tracing analysis for verification of the asymptotic equations.

7.1 Asymmetric bifurcation

Consider an asymmetric bifurcation with $K_2 = 1.0$ and $K_3 = 0.0$. The coefficients at the hilltop point of the perfect system are computed as

$$\begin{aligned} &V_{,111} = -2.0, \quad V_{,222} = -0.61564, \quad V_{,112} = -9.7752 \times 10^{-5} \simeq 0, \\ &V_{,122} = 0, \quad V_{,11\Lambda} = 0, \quad V_{,1111} = 1.4495 \times 10^{-4}, \\ &V_{,2222} = -5.8122 \times 10^{-4}, \quad V_{,1122} = 2.9025 \times 10^{-3}, \quad V_{,2\Lambda} = -1000.0 \end{aligned} \tag{78}$$

and

$$C_\alpha = 1.4512 \times 10^{-4} > 0 \tag{79}$$

The coefficients for the minor imperfection in the direction of $\boldsymbol{\Phi}^{\mathrm{L}}$ are obtained as

$$V_{,2\xi} = 140.46, \quad V_{,11\xi} = -0.20071 \tag{80}$$

The limit point load and bifurcation load of the fundamental path of imperfect systems are obtained from (37) and (38), respectively, as

$$\widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Lim}} = 0.14046\xi, \quad (\beta\xi < 0) \tag{81a}$$

$$\widetilde{\Lambda}^{\mathrm{c}}_{\mathrm{Bif}} = -0.28526\xi, \quad (\beta\xi > 0) \tag{81b}$$

where

$$\beta = 691.51 > 0 \tag{82}$$

The asymptotic relations for Type I solution are plotted in solid lines in Fig. 9, which show good agreement with the results by path-tracing analysis indicated by '+' marks.

For the major imperfection in the direction of bifurcation mode $\boldsymbol{\Phi}^{\mathrm{B}}$, the coefficient is computed as

$$V_{,1\xi} = 64.513 \tag{83}$$

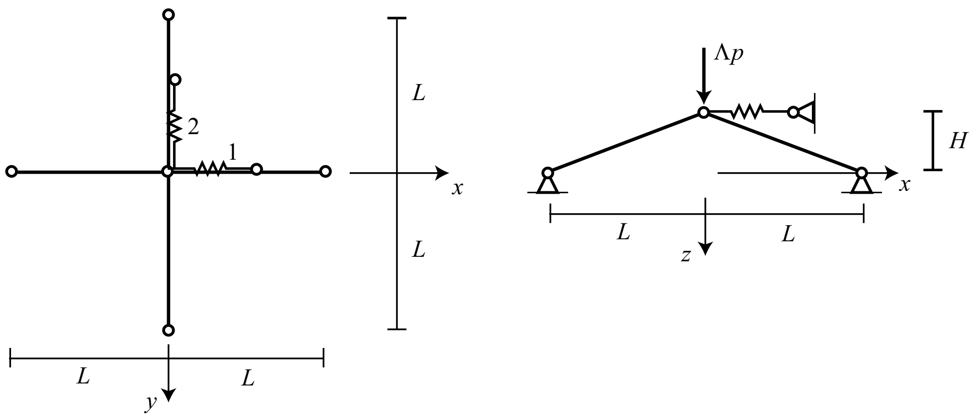

Fig. 8. Four-bar truss tent with nonlinear spring.

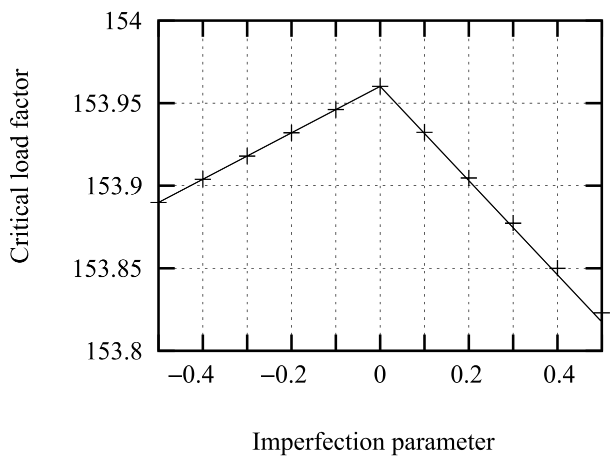

Fig. 9. Imperfection sensitivity for symmetric minor imperfections: asymmetric bifurcation.

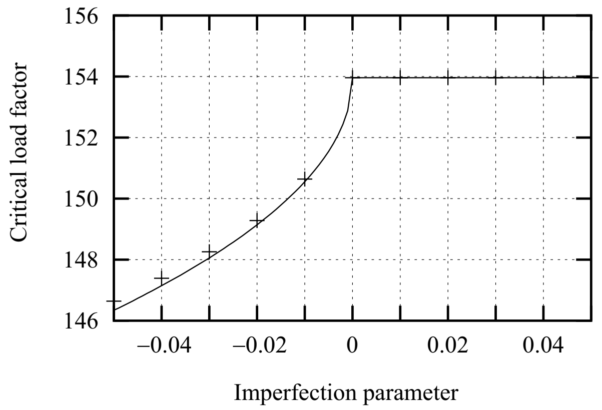

Fig. 10. Imperfection sensitivity for major imperfections: asymmetric bifurcation.

The Type I and II solutions are obtained from (48c) and (50c), respectively, as

$$\text{Type I}: \quad \widetilde{\Lambda}^{\mathrm{c}} = -34.073(-\xi)^{\frac{1}{2}}, \quad (\xi < 0) \tag{84a}$$

$$\text{Type II}: \quad \widetilde{\Lambda}^{\mathrm{c}} = -7.8281 \times 10^{-7} \xi^{\frac{3}{2}}, \quad (\xi > 0) \tag{84b}$$

The asymptotic relations are plotted in solid lines in Fig. 10, which show good agreement with the results by path-tracing analysis indicated by '+' marks.

7.2 Unstable symmetric bifurcation

Consider an unstable symmetric bifurcation point with $K_2 = 0$ and $K_3 = -0.01$. The coefficients at the hilltop point of the perfect system are computed as

$$\begin{aligned} &V_{,111} = 0, \quad V_{,112} = -9.8849 \times 10^{-5} \simeq 0 \\ &V_{,1111} = -5.9855 \times 10^{-2} \end{aligned} \tag{85}$$

The values of other parameters are the same as the case of asymmetric bifurcation.

The critical points on the fundamental path for minor imperfection is the same as the case of the asymmetric bifurcation as shown in Fig. 9. For the major imperfection in the direction of bifurcation mode $\mathbf{\Phi}^{\mathrm{B}}$, the coefficient is computed as

$$V_{,1\xi} = 64.513 \tag{86}$$

The Type I and II solutions are obtained from (74c) and (75c), respectively, as

$$\text{Type I}: \quad \widetilde{\Lambda}^{\mathrm{c}} = -13.880 \xi^{\frac{2}{3}} \tag{87a}$$

$$\text{Type II}: \quad \widetilde{\Lambda}^{\mathrm{c}} = -2.2468 \times 10^{-7} \xi^{\frac{4}{3}} \tag{87b}$$

Note that only Type I solution has practical importance, because Type II solution exists on an aloof path. The asymptotic relations are plotted in solid lines in Fig. 11, which show good agreement with the results by path-tracing analysis indicated by '+' marks.

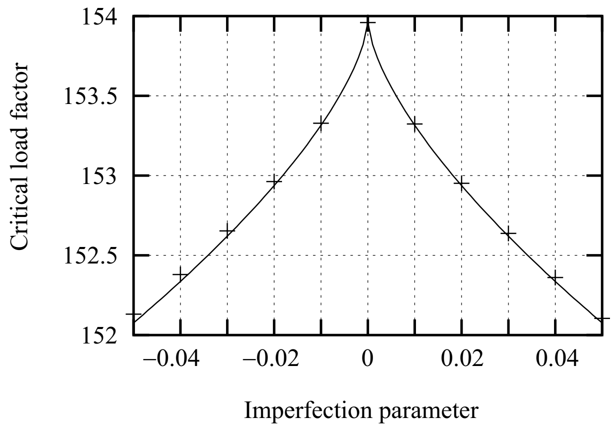

Fig. 11. Imperfection sensitivity for major imperfections: unstable-symmetric bifurcation.

8 Conclusions

Imperfection sensitivity has been investigated for a degenerate hilltop point, where a degenerate bifurcation point exists at a limit point. The degenerate hilltop point is first defined on the basis of the coefficients of the asymptotic expansion of the total potential energy with respect to the active coordinates and the load factor. The degenerate hilltop point is characterized as the coincident critical point such that the derivative of the eigenvalue corresponding to the bifurcation point vanishes along the fundamental path of the perfect system.

Imperfect behaviors are next investigated considering minor and major imperfections for unstable-symmetric and asymmetric bifurcation points. By means of the power series expansion method with the assist of the concept of the order of a solution that is a strong and insightful tool, imperfection sensitivity laws of coincident critical points, including degenerate hilltop branching points, have successfully been derived.

A systematic procedure has been presented for asymptotic sensitivity analysis based on enumeration of vertices of a convex region defined by linear inequality constraints on the orders of the generalized coordinates and the load factor with respect to the imperfection parameter. The symbolic computation package has emerged as a powerful and robust tool for derivation of the inequality constraints. The enumeration method presented in this paper can be effectively applied to any coincident critical points, where the consistent set of orders cannot be enumerated by intuition.

Appendix 1: Enumeration of feasible set of orders for asymptotic expansion

A method is presented below for enumerating the feasible set of orders for asymptotic expansion.

Imperfection sensitivity laws defining the relations among q_1, q_2 and $\widetilde{\Lambda}$ at the critical points of imperfect systems can be derived from the bifurcation equations and the criticality condition $\det \mathbf{S} = 0$. For example, these equations are derived as (30), (31) and $\det \mathbf{S} = 0$ with (32) for a hilltop point with asymmetric bifurcation. As is seen, these equations form a set of two quadratic polynomials and one fourth-order polynomial that cannot be solved analytically.

Therefore, in the standard approaches developed so far, the solution set is estimated by assuming the orders of q_1, q_2 and $\widetilde{\Lambda}$ with respect to ξ. However, it is still difficult to find all the possible and consistent set of orders by an intuitive approach. In this section, we present a systematic approach for enumerating all the consistent sets of orders.

The bifurcation equations and the criticality condition are symbolically written as

$$A(q_1, q_2, \widetilde{\Lambda}, \xi) \equiv \frac{\partial V}{\partial q_1} = 0 \tag{A.1a}$$

$$B(q_1, q_2, \widetilde{\Lambda}, \xi) \equiv \frac{\partial V}{\partial q_2} = 0 \tag{A.1b}$$

$$D(q_1, q_2, \widetilde{\Lambda}, \xi) \equiv \det \mathbf{S} = 0 \tag{A.1c}$$

which are polynomials of q_1, q_2, $\widetilde{\Lambda}$ and ξ.

The orders of q_1, q_2 and $\widetilde{\Lambda}$ expressed in terms of the powers of ξ are denoted by H_1, H_2 and H_Λ, respectively; i.e.,

$$q_1 = O(\xi^{H_1}), \quad q_2 = O(\xi^{H_2}), \quad \widetilde{\Lambda} = O(\xi^{H_\Lambda}) \tag{A.2}$$

where $O(\,\cdot\,)$ denotes the order of the term in the parentheses.

Let R_A, R_B and R_D denote the lowest orders relative to ξ of the terms in A, B and D, respectively. For example, the order of $V_{,11\Lambda} q_1 \widetilde{\Lambda}$ for $q_1 = O(\xi^{\frac{1}{2}})$ and $\widetilde{\Lambda} = O(\xi^{\frac{3}{2}})$ is $1/2 + 3/2 = 2$. The sets of orders of the power of $(\xi, q_1, q_2, \widetilde{\Lambda})$ in the ith term of the polynomials A, B and D, respectively, are denoted by $(A^i_\xi, A^i_1, A^i_2, A^i_\Lambda)$, $(B^i_\xi, B^i_1, B^i_2, B^i_\Lambda)$ and $(D^i_\xi, D^i_1, D^i_2, D^i_\Lambda)$. For example, the term $\frac{1}{2} V_{,1122}(q_1)^2 q_2$ in (31) leads to $(B^i_\xi, B^i_1, B^i_2, B^i_\Lambda) = (0, 2, 1, 0)$.

Then the feasible solutions are characterized by the following inequalities:

$$A^i_\xi + A^i_1 H_1 + A^i_2 H_2 + A^i_\Lambda H_\Lambda \geq R_A, \quad (i = 1, \ldots, N_A) \tag{A.3a}$$

$$B^i_\xi + B^i_1 H_1 + B^i_2 H_2 + B^i_\Lambda H_\Lambda \geq R_B, \quad (i = 1, \ldots, N_B) \tag{A.3b}$$

$$D^i_\xi + D^i_1 H_1 + D^i_2 H_2 + D^i_\Lambda H_\Lambda \geq R_D, \quad (i = 1, \ldots, N_D) \tag{A.3c}$$

where N_A, N_B and N_D are the numbers of terms in A, B and D, respectively. The inequalities (A.3) are automatically generated by a symbolic computation package Maple 11 [30].

For example, consider a simple symmetric bifurcation point governed by the bifurcation equation and criticality condition as

$$V_{,\xi}\xi + V_{,11\Lambda} q_1 \widetilde{\Lambda} + V_{,1\Lambda\Lambda} \widetilde{\Lambda}^2 + V_{,1111}(q_1)^3 = 0 \tag{A.4a}$$

$$V_{,11\Lambda} \widetilde{\Lambda} + 3V_{,1111}(q_1)^2 = 0 \tag{A.4b}$$

In this case, $N_A = 4$, $N_B = 0$, $N_D = 2$, and (A.3a) and (A.3c) are written as

$$1 \geq R_A, \quad H_1 + H_\Lambda \geq R_A, \quad 2H_\Lambda \geq R_A, \quad 3H_1 \geq R_A \tag{A.5a}$$

$$H_\Lambda \geq R_D, \quad 2H_1 \geq R_D \tag{A.5b}$$

Table A.1. Vertices of the feasible region of the orders for minor imperfection to hilltop point with asymmetric bifurcation.

vertex	H_1	H_2	H_Λ	M_A	M_B	M_D	
1	1	1/2	1	4	3	4	accept
2	1/4	1/2	1	1	4	2	reject because $M_A = 1$
3	1/2	1	1	1	2	4	reject because $M_A = 1$
4	1	1	1	3	2	8	accept

in which (A.3b) is nonexistent. From (A.5), the two sets $(H_1, H_\Lambda, R_A, R_D) = (1/3, 2/3, 1, 2/3)$ and $(1/2, 1/2, 1, 1/2)$ are obtained by the vertex enumeration of the convex region defined by the linear inequalities. However, the latter leads to $\widetilde{\Lambda} = 0$ by (A.4b), and is not physically feasible. Therefore, from $(H_1, H_\Lambda) = (1/3, 2/3)$ in the first set, we obtain the following well-known relation for the two-third power law:

$$q_1^{\mathrm{c}} = O(\xi^{\frac{1}{3}}), \quad \widetilde{\Lambda}^{\mathrm{c}} = O(\xi^{\frac{2}{3}}) \tag{A.6}$$

This result agrees with the well-known formula [22] and the result by Newton-Polygon approach in [28].

Let M_A, M_B and M_D denote the numbers of inequalities satisfied in equality for (A.3a), (A.3b) and (A.3c), respectively. For the consistent set of orders, at least two inequalities should be satisfied in equality for each of (A.3a), (A.3b) and (A.3c); i.e., M_A, M_B and M_D should not be less than two, because some of q_1, q_2, $\widetilde{\Lambda}$ and ξ may vanish if only one inequality is satisfied in equality and only one term remains in (A.1a), (A.1b) or (A.1c) as the lowest-order term. For example, if $1 > R_A$, $H_1 + H_\Lambda = R_A$, $2H_\Lambda > R_A$ and $3H_1 > R_A$ in (A.5a), only the second term remains as the lowest-order term in (A.4a), and q_1 or $\widetilde{\Lambda}$ should vanish by (A.4a).

Therefore, we have at least six equations for six variables R_A, R_B, R_D, H_1, H_2 and H_Λ. Since all inequalities in (A.3) are linear with respect to the variables, the solutions exist at the vertices of the feasible region, and the solutions can be found by vertex enumeration of the region defined by linear inequalities.

In the following, we use the library cdd+ Ver. 0.76 [32, 33] for the vertex enumeration. Then the vertices that do not have a vanishing variable are chosen as the asymptotic solutions.

For the bifurcation path of the imperfect systems corresponding to minor imperfection, the inequalities (A.3) for the feasible region of the orders H_1, H_2 and H_Λ of q_1, q_2 and $\widetilde{\Lambda}$ at the critical points are generated, respectively, from (30), (31) and $\det \mathbf{S} = 0$ with (32) as

$$2H_1 \geq R_A, \quad H_1 + H_\Lambda \geq R_A, \quad 1 + H_1 \geq R_A, \quad H_1 + 2H_2 \geq R_A \tag{A.7}$$

$$1 \geq R_B, \quad H_\Lambda \geq R_B, \quad 2H_2 \geq R_B, \quad 2H_1 + H_2 \geq R_B \tag{A.8}$$

$$\left\{ \begin{array}{l} H_1 + H_2 \geq R_D, \quad H_1 + H_\Lambda \geq R_D, \quad 1 + H_1 \geq R_D, \quad 3H_1 \geq R_D, \\ H_2 + H_\Lambda \geq R_D, \quad 2H_\Lambda \geq R_D, \quad 1 + H_\Lambda \geq R_D, \quad 2H_1 + H_\Lambda \geq R_D, \\ 1 + H_2 \geq R_D, \quad 2 \geq R_D, \quad 1 + 2H_1 \geq R_D, \quad 3H_2 \geq R_D, \\ 2H_2 + H_\Lambda \geq R_D, \quad 1 + 2H_2 \geq R_D, \quad 2H_1 + H_2 + H_\Lambda \geq R_D, \quad 3H_1 + H_2 \geq R_D, \\ 2H_1 + 2H_\Lambda \geq R_D, \quad 3H_1 + H_\Lambda \geq R_D, \quad 4H_1 \geq R_D, \quad 2H_1 + 2H_2 \geq R_D \end{array} \right. \tag{A.9}$$

Table A.2. Vertices of the feasible region of the orders for major imperfection to hilltop point with asymmetric bifurcation.

vertex	H_1	H_2	H_Λ	M_A	M_B	M_D	
1	1/2	1	1	2	1	3	reject because $\widetilde{\Lambda} = 0$
2	1/2	1/2	1/2	3	1	4	reject because $\widetilde{\Lambda} = 0$
3	1/2	1/4	1/2	4	2	3	accept
4	1/2	3/4	3/2	2	3	1	reject because $M_D = 1$
5	1/2	1	3/2	3	2	2	accept
6	1/3	2/3	4/3	1	4	2	reject because $\xi = 0$
7	1	1	1	1	1	5	reject because $\xi = 0$
8	4/3	2/3	4/3	1	2	4	reject because $\xi = 0$
9	2/3	4/3	4/3	1	1	4	reject because $\xi = 0$
10	2/3	4/3	5/3	1	2	3	reject because $\xi = 0$

Table A.3. Vertices of the feasible region of the orders for minor imperfection to hilltop point with unstable-symmetric bifurcation.

vertex	H_1	H_2	H_Λ	M_A	M_B	M_D	
1	1/2	1/2	1	4	3	4	accept
2	1/4	1/2	1	1	4	2	reject because $M_A = 1$
3	1/2	1	1	3	2	8	accept

The enumerated vertices are listed in Table A.1. The first vertex $(H_1, H_2, H_\Lambda) = (1, 1/2, 1)$ corresponds to the bifurcation point at the intersection with the fundamental path. From the fourth set $(H_1, H_2, H_\Lambda) = (1, 1, 1)$, we have the location of a limit point on the bifurcation or aloof path as (42). The second and third vertices are rejected, because $M_A = 1$ that leads to $q_1 = 0$.

For major imperfection, the inequalities (A.3) for the feasible region of the orders H_1, H_2 and H_Λ of q_1, q_2 and $\widetilde{\Lambda}$ are generated, respectively, from (30), (31) and $\det \mathbf{S} = 0$ with (32) as

$$1 \geq R_A, \quad 2H_1 \geq R_A, \quad H_1 + H_\Lambda \geq R_A, \quad H_1 + 2H_2 \geq R_A \tag{A.10}$$

$$H_\Lambda \geq R_B, \quad 2H_2 \geq R_B, \quad 1 + H_1 \geq R_B, \quad 2H_1 + H_2 \geq R_B \tag{A.11}$$

$$\left\{ \begin{array}{l} H_1 + H_2 \geq R_D, \quad H_1 + H_\Lambda \geq R_D, \quad 3H_1 \geq R_D, \quad H_2 + H_\Lambda \geq R_D, \\ 2H_\Lambda \geq R_D, \quad 2H_1 + H_\Lambda \geq R_D, \quad 3H_2 \geq R_D, \quad 2H_2 + H_\Lambda \geq R_D, \\ 2 \geq R_D, \quad 1 + H_1 + H_2 \geq R_D, \quad 1 + H_1 + H_\Lambda \geq R_D, \quad 1 + 2H_1 \geq R_D, \\ 2H_1 + H_2 + H_\Lambda \geq R_D, \quad 3H_1 + H_2 \geq R_D, \quad 2H_1 + 2H_\Lambda \geq R_D, \\ 3H_1 + H_\Lambda \geq R_D, \quad 4H_1 \geq R_D, \quad 2H_1 + 2H_2 \geq R_D \end{array} \right. \tag{A.12}$$

Then the vertices of the feasible region are found as shown in Table A.2.

The vertices are also enumerated for hilltop point with unstable-symmetric bifurcation in the similar manner as the case with asymmetric bifurcation. The vertices for minor imperfection and major imperfection are listed in Tables A.3 and A.4, respectively.

Table A.4. Vertices of the feasible region of the orders for major imperfection to hilltop point with unstable-symmetric bifurcation.

vertex	H_1	H_2	H_Λ	M_A	M_B	M_D	
1	1/3	2/3	2/3	3	1	5	reject because $M_B = 1$
2	1/3	1/3	2/3	4	2	3	accept
3	3/5	4/5	8/5	1	3	2	reject because $M_A = 1$
4	1/2	1	2/3	1	2	3	reject because $M_A = 1$
5	1/2	1	1	1	1	6	reject because $M_A = 1$
6	2/3	2/3	4/3	1	2	4	reject because $M_A = 1$
7	1/3	2/3	4/3	2	4	2	accept

Appendix 2: Details of bifurcation equations and criticality conditions

The detailed expressions are presented below for the bifurcation equations and the criticality condition for each specific case in Sections 5 and 6.

Asymmetric bifurcation

Tyle I) solution for major imperfection

$$V_{,1\xi}\xi + \frac{1}{2}V_{,111}(q_1)^2 + V_{,11\Lambda}q_1\widetilde{\Lambda} + \frac{1}{2}V_{,1122}q_1(q_2)^2 = 0 \tag{A.13}$$

$$V_{,2\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,222}(q_2)^2 = 0 \tag{A.14}$$

$$V_{,111}q_1 + V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \tag{A.15}$$

Tyle II) solution for major imperfection

$$V_{,1\xi}\xi + \frac{1}{2}V_{,111}(q_1)^2 = 0 \tag{A.16}$$

$$V_{,2\Lambda}\widetilde{\Lambda} + V_{,12\xi}q_1\xi = 0 \tag{A.17}$$

$$V_{,222}q_2 + \frac{1}{2}V_{,1122}(q_1)^2 = 0 \tag{A.18}$$

Unstable-symmetric bifurcation

Minor imperfection

$$V_{,11\Lambda}\widetilde{\Lambda} + V_{,11\xi}\xi + \frac{1}{6}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \tag{A.19}$$

$$V_{,2\Lambda}\widetilde{\Lambda} + V_{,2\xi}\xi + \frac{1}{2}V_{,222}(q_2)^2 = 0 \tag{A.20}$$

$$V_{,222}q_2\Big(V_{,11\Lambda}\widetilde{\Lambda} + V_{,11\xi}\xi + \frac{1}{2}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2\Big) = 0 \tag{A.21}$$

Tyle I) solution for major imperfection

$$V_{,1\xi}\xi + V_{,11\Lambda}q_1\widetilde{\Lambda} + \frac{1}{6}V_{,1111}(q_1)^3 + \frac{1}{2}V_{,1122}q_1(q_2)^2 = 0 \tag{A.22}$$

$$V_{,2\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,222}(q_2)^2 = 0 \tag{A.23}$$

$$V_{,11\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,1111}(q_1)^2 + \frac{1}{2}V_{,1122}(q_2)^2 = 0 \tag{A.24}$$

Tyle II) solution for major imperfection

$$V_{,1\xi}\xi + \frac{1}{6}V_{,1111}(q_1)^3 = 0 \tag{A.25}$$

$$V_{,2\Lambda}\widetilde{\Lambda} + \frac{1}{2}V_{,222}(q_2)^2 + V_{,12\xi}q_1\xi + \frac{1}{2}V_{,1122}(q_1)^2q_2 = 0 \tag{A.26}$$

$$V_{,222}q_2 + \frac{1}{2}V_{,1122}(q_1)^2 = 0 \tag{A.27}$$

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