Stability Analysis of Cable–Bar Structures by Inverse-Power Method for Eigenvalue Analysis with Penalization

M. Ohsaki[1] * , Y. Kanno[2]

[1] *Department of Architecture and Architectural Engineering, Kyoto University, Kyoto-Daigaku Katsura, Nishikyo, Kyoto, 615-8540, Japan,*
[2] *Department of Mathematical Informatics, University of Tokyo, Tokyo 113-8656, Japan*

Abstract

A numerical method is presented for stability analysis of cable–bar structures. An optimization problem is formulated to find the minimum value of the incremental total potential energy that depends on the direction of the incremental displacements. The penalty method with slack variables is used for representing the discontinuity in member stiffness. The tangent stiffness matrix is shifted to be positive definite so that the minimum of its quadratic form is found by the inverse-power method. It is shown in the numerical examples that the minimum value of the incremental potential energy and the associated displacement increments can be found with good accuracy in about 10 steps of iteration.

Key words: stability; cable–bar structure; potential energy; penalty method

* Corresponding author. Tel: +81-75-383-2901 Fax: +81-75-383-2972.
Email addresses: ohsaki@archi.kyoto-u.ac.jp (M. Ohsaki[1]), kanno@mist.i.u-tokyo.ac.jp (Y. Kanno[2]).

1 Introduction

Stability analysis of elastic structures is a rather established field of research, and there have been numerous number of papers on numerical techniques for detecting instability of finite dimensional structures (e.g., Riks (1998); Wriggers and Simo (1990)).

Based on Liapunov's direct method (Salle and Lefscetz, 1961; Pignataro *et al.*, 1991), stability of an elastic conservative system is defined by isolated local minimum of the total potential energy. For cases in which the potential energy is twice differentiable with respect to the displacements, the stability of a given equilibrium state in finite deformation is defined by the positive definiteness of the tangent stiffness matrix (stability matrix) (Thompson and Hunt, 1973).

A cable–bar structure consists of the cable members that can transmit tensile forces only and the bars that can transmit both compressive and tensile forces. A bar that transmits compressive force only is called a strut. A structure that consists of cables and struts is called a tensegrity structure. Since the cable member has no flexural stiffness, tensegrity structures are usually stabilized by introducing prestresses to maintain self-equilibrium state. In this paper, we assume that a bar can transmit both tensile and compressive forces.

Inability of the cable to transmit compressive force leads to discontinuity of the tangent stiffness matrix (Panagiotopoulos, 1976). The authors investigated the minimum complementary principle for cable networks undergoing large deformation (Kanno and Ohsaki, 2003). The first author presented stability conditions for cable-bar structures (Ohsaki and Zhang, 2006; Zhang and Ohsaki, 2007).

Discontinuity in tangent stiffness matrix is also observed in contact problems and elastoplastic material models. For an elastoplastic structure, the uniqueness of equilibrium state is defined by the positive definiteness of the inloading tangent stiffness matrix (Hill, 1958), while its stability is defined based on the directional stability (Bigoni, 2000). For frictional contact problem with a non-associated friction law, for which a potential energy cannot be defined, a method has been developed for stability analysis also based on directional stability (Costa *et al.*, 2004).

Choong and Hangai (1993) presented an iterative approach for bifurcation analysis of beams and arches with unilateral supports. Tschöpe *et al.* (2003) developed an iterative approach to direct computation of the critical point involving frictionless contact conditions. Villagio (1979) formulated the buckling analysis problem of a beam with unilateral supports by minimization of the Rayleigh quotient, but did not present a numerical algorithm.

In this paper, we present a numerical method for stability analysis of cable–bar structures. The total potential energy is a smooth function of the nodal displacements, but is not twice differentiable; i.e., the tangent stiffness matrix depends on the direction of the displacement increment. However, the total potential energy satisfies the assumption for the stability theorem by Liapunov, and the stability of the given equilibrium state is defined by the isolated local minimum of the total potential energy.

This paper is organized as follows. Stability conditions are briefly summarized in Section 2. In Section 3, an optimization problem is formulated to find the minimum of the quadratic form of the tangent stiffness matrix that depends on the direction of the incremental displacements. A slack variable is used for representing the discontinuity in member stiffness. In Section 4, the tangent stiffness matrix is shifted to be positive definite and the constraints are incorporated by penalty approach so that its minimum incremental potential energy is found by the inverse-power method. This way, the difficulty due to nonconvexity of the potential energy at an unstable equilibrium state is successfully overcome. The conditions satisfied by the optimal solution and the convergence property are investigated in Section 5. It is shown in the numerical examples in Section 6 that the minimum incremental potential energy and the associated displacement increments can be found with good accuracy in about 10 steps of iteration.

2 Stability conditions

Consider a cable–bar structure consisting of cable members that transmit tensile forces only, and the bars that can transmit both compressive and tensile forces. We use the assumption of small strain, and the deformation before reaching the equilibrium state for which the stability is investigated is assumed to be small.

Let $\mathbf{u} \in \mathbb{R}^n$ denote an admissible incremental displacement vector satisfying the kinematic boundary conditions, where n is the number of degrees of freedom of displacements. We assume, for simplicity, that all the boundary conditions are homogeneous, and the components corresponding to fixed degree-of-freedom have been removed before constructing $\mathbf{u}$; i.e., any vector $\mathbf{u} \in \mathbb{R}^n$ is kinematically admissible. The vector of incremental member extensions is denoted by $\mathbf{d} \in \mathbb{R}^s$, where s is the number of members including cables and bars. Note that the slack cables are excluded *a priori*, because they have no effect on the structural properties under infinitesimal incremental displacements. The relation between $\mathbf{u}$ and $\mathbf{d}$ is defined by using the constant matrix $\mathbf{H} \in \mathbb{R}^{s \times n}$ as

$$\mathbf{d} = \mathbf{H}\mathbf{u} \tag{1}$$

In the following, all vectors are column vectors and the component is indicated by a subscript.

Let k_i denote the extensional stiffness of the ith member. If the ith member is a bar, it has a linear force-extension relation with the stiffness k_i. If the ith member is a cable, k_i is the stiffness in tensile state. The set of indices of the cables that has zero extension at the equilibrium state is denoted by $\mathcal{I}$. The ith component of $\mathbf{d}$ is denoted by d_i. The relation between d_i and the incremental force q_i is written as

$$q_i = \begin{cases} 0 \text{ for } i \in \mathcal{I} \text{ and } d_i < 0 \\ d_i k_i \text{ for other cases} \end{cases} \tag{2}$$

where the axial force and extension are defined to be positive in tensile state.

The stability of a static equilibrium state is defined with the use of dynamical system based on Liapunov's direct method (Salle and Lefscetz, 1961; Pignataro *et al.*, 1991). Let $\dot{\mathbf{u}}$ denote the velocity vector and define the state variable vector $\mathbf{x}$ by $\mathbf{x} = (\mathbf{u}^\top, \dot{\mathbf{u}}^\top)^\top$. The total energy $R(\mathbf{x})$, which is the sum of the potential energy and the kinetic energy, can be chosen as the Liapunov function satisfying

C1: $R(\mathbf{x})$ and its first derivative are continuous functions of $\mathbf{x}$.
C2: $R(\mathbf{0}) = 0$.
C3: $R(\mathbf{0})$ is an isolated minimum of $R(\mathbf{x})$.
For a moderately dumped system with positive definite damping matrix, the origin $\mathbf{x} = \mathbf{0}$ is an isolated minimum of the kinetic energy, and the equilibrium state corresponding to $\mathbf{x} = \mathbf{0}$ is stable if the incremental total potential energy $\Pi(\mathbf{u})$ measured from the current equilibrium state attains an isolated minimum at $\mathbf{u} = \mathbf{0}$.

The only one difference between a conventional conservative system and the cable–bar structure is that the constitutive relation is given as (2) for the latter. Although the stiffness of member i depends on the sign of d_i, the strain energy $q_i d_i/2$ and its derivative with respect to d_i are continuous functions of d_i. Therefore, the condition C1 is satisfied, and the current equilibrium state $\mathbf{u} = \mathbf{0}$ is stable if $\Pi(\mathbf{u})$ attains an isolated local minimum at $\mathbf{u} = \mathbf{0}$.

Since the first derivative of $\Pi(\mathbf{u})$ with respect to $\mathbf{u}$ vanishes from the equilibrium conditions, the stability is defined by the quadratic term of $\Pi(\mathbf{u})$. Suppose that the direction of the incremental displacements $\mathbf{u}$ is given. The tangent stiffness matrix consistent to (2) is denoted by $\hat{\mathbf{K}}(\mathbf{u}) \in \mathbb{R}^{n\times n}$. The twice of the quadratic term of $\Pi(\mathbf{u})$ is written as

$$\hat{V}(\mathbf{u}) = \mathbf{u}^\top \hat{\mathbf{K}}(\mathbf{u})\mathbf{u} \tag{3}$$

The equilibrium state is stable if $\hat{V}(\mathbf{u}) = \mathbf{u}^\top \hat{\mathbf{K}}(\mathbf{u})\mathbf{u} > 0$ for any admissible $\mathbf{u}$. On the contrary, the structure is unstable if there exists an admissible $\mathbf{u}$ satisfying $\mathbf{u}^\top \hat{\mathbf{K}}(\mathbf{u})\mathbf{u} < 0$.

3 Minimization of incremental potential energy

Stability is investigated by minimizing $\hat{V}(\mathbf{u})$ with respect to $\mathbf{u}$. Consider the following optimization problem:

$$\begin{aligned} \mathrm{P1:}\ \ & \text{minimize} \quad \hat{V}(\mathbf{u}) = \mathbf{u}^\top \hat{\mathbf{K}}(\mathbf{u})\mathbf{u} & \text{(4a)} \\ & \text{subject to} \quad N(\mathbf{u}) = 1 & \text{(4b)} \end{aligned}$$

where the constraint (4b) is given for preventing convergence to the trivial solution $\mathbf{u} = \mathbf{0}$ for the case where $\hat{V}(\mathbf{u})$ is positive for any $\mathbf{u}$ ($\mathbf{u} \neq \mathbf{0}$). In the following, we use the quadratic constraint as $N(\mathbf{u}) = \mathbf{u}^\top\mathbf{u} = 1$.

If the optimal value of P1 is positive, then the equilibrium state is stable. However, the constraint $\mathbf{u}^\top\mathbf{u} = 1$ is not convex, and the objective function is nonconvex if the equilibrium state is unstable. Therefore, the global optimality of the solution of P1 obtained by a nonlinear programming cannot be guaranteed.

The incremental extension d_i of the ith member is decomposed using the slack variables d_i^+ and d_i^- as

$$d_i = d_i^+ - d_i^-, \quad d^+ \geq 0, \quad d_i^- \geq 0, \quad d_i^+ d_i^- = 0, \quad (i = 1, \dots, s) \tag{5}$$

Since $k_i > 0$, the complementarity condition $d_i^+ d_i^- = 0$ with $d^+ \geq 0$ in (5) is automatically satisfied by minimizing the quadratic term $V(\mathbf{u}, \mathbf{d}^+)$ of the incremental potential energy, which is defined as

$$V(\mathbf{u}, \mathbf{d}^+) = \sum_{i \in \mathcal{I}} (d_i^+)^2 k_i + \mathbf{u}^\top \mathbf{K}^+ \mathbf{u} \tag{6}$$

where $\mathbf{K}^+ \in \mathbb{R}^{n \times n}$ is the tangent stiffness matrix consisting of the cables in tensile state and the bars. Therefore, (5) is written as

$$d_i^+ - d_i \geq 0, \quad (i = 1, \ldots, s) \tag{7}$$

The first term in (6) corresponds to the strain energy of the cables with zero extension. The second term is the strain energy of the bars and the cables in tensile state. The equilibrium state is stable if $V(\mathbf{u}, \mathbf{d}^+)$ is positive for any admissible set of $\mathbf{u}$ and $\mathbf{d}^+$ satisfying (1) and (5).

Let $\mathbf{h}_i^\top$ denote the ith row of $\mathbf{H}$. Then from (1), (7) is rewritten as

$$\mathbf{e}^i \mathbf{d}^+ - \mathbf{h}_i \mathbf{u} \geq 0, \quad (i = 1, \ldots, s) \tag{8}$$

where the elements in $\mathbf{e}^i \in \mathbb{R}^m$ are 0 except 1 in the ith element.

Let m denote the number of members in $\mathcal{I}$ which are numbered for simplicity as $1, \ldots, m$. A matrix $\mathbf{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ and a vector $\mathbf{t} \in \mathbb{R}^{n+m}$ are defined as

$$\mathbf{A} = \begin{pmatrix} \mathbf{K}^+ & \mathbf{O} \\ \mathbf{O} & \mathrm{diag}(k_1, \ldots, k_m) \end{pmatrix} \tag{9}$$

$$\mathbf{t} = (u_1, \ldots, u_n, d_1^+, \ldots, d_m^+)^\top \tag{10}$$

where $\mathrm{diag}(k_1, \ldots, k_m)$ is a diagonal matrix.

Define $\mathbf{g}_i$ as

$$\mathbf{g}_i = (\mathbf{h}_i^\top, -\mathbf{e}^{i\top})^\top \tag{11}$$

Hence, (8) is written as

$$\mathbf{g}_i^\top \mathbf{t} \leq 0, \quad (i = 1, \ldots, s) \tag{12}$$

Then P1 is then rewritten as

$$\begin{aligned} \text{P2}: \ \text{minimize} \quad & V(\mathbf{t}) = \mathbf{t}^\top \mathbf{A} \mathbf{t} && \text{(13a)} \\ \text{subject to} \quad & \mathbf{g}_i^\top \mathbf{t} \leq 0, \quad (i = 1, \ldots, m) && \text{(13b)} \\ & \mathbf{u}(\mathbf{t})^\top \mathbf{u}(\mathbf{t}) = 1 && \text{(13c)} \end{aligned}$$

The structure is stable if $V(\mathbf{t})$ is positive at the optimal solution of P2. Note again that the constraint (13c) is given to prevent obtaining the degenerate solution $\mathbf{t} = \mathbf{0}$ for the case where the minimum of $V(\mathbf{t})$ for $\mathbf{t} \neq \mathbf{0}$ is positive. Since $\mathbf{u} \neq \mathbf{0}$ for $\mathbf{d} \neq \mathbf{0}$, we use the quadratic constraint $\mathbf{t}^\top \mathbf{t} = 1$ instead of (13c). Then the sign of the optimal value of P2 coincides with that of P3 defined as

$$\begin{aligned} \text{P3}: \ \text{minimize} \quad & V(\mathbf{t}) = \mathbf{t}^\top \mathbf{A} \mathbf{t} && \text{(14a)} \\ \text{subject to} \quad & \mathbf{g}_i^\top \mathbf{t} \leq 0, \quad (i = 1, \ldots, m) && \text{(14b)} \\ & \mathbf{t}^\top \mathbf{t} = 1 && \text{(14c)} \end{aligned}$$

The ith eigenvalue of the symmetric matrix $\mathbf{A}$ is denoted by λ_i^A ($\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_{n+m}^A$). If $\mathbf{A}$ is positive definite, then the equilibrium state is stable, and it is easily confirmed that the optimal value of P3 is positive. Therefore, in the following, we consider the case where $\mathbf{A}$ is not positive definite; i.e., $\lambda_1^A \leq 0$.

Let $\mathbf{I} \in \mathbb{R}^{(n+m)\times(n+m)}$ denote an identity matrix, and for a sufficiently large λ^* ($> |\lambda_1^A|$), define $\mathbf{A}^*$ by

$$\mathbf{A}^* = \mathbf{A} + \lambda^*\mathbf{I} \tag{15}$$

Then the eigenvalues of $\mathbf{A}^* \in \mathbb{R}^{(n+m)\times(n+m)}$ satisfy $\lambda_i^{A*} = \lambda_i^A + \lambda^* > 0$ ($i = 1, \dots, m+n$), and $\mathbf{A}$ and $\mathbf{A}^*$ share the same set of eigenvectors.

Accordingly, the structure is stable if the optimal value of the following problem P4 is greater than λ^*.

$$\begin{aligned}
\text{P4}:\ & \text{minimize} && V^*(\mathbf{t}) = \mathbf{t}^\top \mathbf{A}^* \mathbf{t} && \text{(16a)}\\
& \text{subject to} && \mathbf{g}_i^\top \mathbf{t} \leq 0, \quad (i = 1, \dots, m) && \text{(16b)}\\
& && \mathbf{t}^\top \mathbf{t} = 1 && \text{(16c)}
\end{aligned}$$

4 Optimization algorithm by using penalty approach

In order to solve P4 by the inverse-power method (Atkinson, 1989), the objective function is converted to $\tilde{V}(\mathbf{t})$ as follows by incorporating the constraint (16b) as the penalty term:

$$\tilde{V}(\mathbf{t}) = \mathbf{t}^\top \mathbf{A}^* \mathbf{t} + \sum_{i=1}^{m} \mu_i (\mathbf{g}_i^\top \mathbf{t})^2 \tag{17}$$

where $\mu_i > 0$ is specified as follows using a positive penalty parameter μ:

$$\begin{aligned}
\mu_i = \mu \quad &\text{for} \quad \mathbf{g}_i^\top \mathbf{t} > 0\\
\mu_i = 0 \quad &\text{for} \quad \mathbf{g}_i^\top \mathbf{t} \leq 0
\end{aligned} \tag{18}$$

Define a matrix $\mathbf{C} \in \mathbb{R}^{(n+m)\times(n+m)}$ as

$$\mathbf{C} = \mathbf{A}^* + \mathbf{P} \tag{19}$$

where

$$\mathbf{P} = \sum_{i=1}^{m} \mu_i \mathbf{g}_i \mathbf{g}_i^\top \tag{20}$$

Note that $\mathbf{C}$ is positive definite by the definition of $\mathbf{A}^*$ and $\mathbf{P}$. Hence, the stability of the structure is detected by solving the following problem:

$$\begin{aligned}
\text{P5}:\ & \text{minimize} && \tilde{V}(\mathbf{t}) = \mathbf{t}^\top \mathbf{C} \mathbf{t} && \text{(21a)}\\
& \text{subject to} && \mathbf{t}^\top \mathbf{t} = 1 && \text{(21b)}
\end{aligned}$$

If $\mathbf{C}$ is constant, P5 is a problem of finding the minimum eigenvalue of a positive definite matrix. However, $\mathbf{C}$ depends on $\mathbf{t}$ through $\mathbf{P}$, but we can iteratively update $\mathbf{C}$ and find the minimum objective value of P5 by the inverse-power method as

Step 1 Specify the constants λ^* and μ.
Step 2 Assign initial value of $\mathbf{t}$.
Step 3 Normalize $\mathbf{t}$ by $\mathbf{t}^\top\mathbf{t} = 1$, and compute $\tilde{V}(\mathbf{t})$.
Step 4 Set $\mu_i = \mu$ for $\mathbf{g}_i^\top\mathbf{t} > 0$; otherwise set $\mu_i = 0$.
Step 5 Compute $\mathbf{C}$.
Step 6 Solve the linear equations $\mathbf{Cy} = \mathbf{t}$ for $\mathbf{y}$ and let $\mathbf{t} \leftarrow \mathbf{y}$.
Step 7 Go to Step 3 if not converged.

5 Optimality conditions and convergence properties

The property of the optimal solution can be investigated by the optimality conditions of P5. Consider first, for comparison purpose, an elastic structure without discontinuity in tangent stiffness matrix denoted by $\mathbf{K}$. Then the stability of the equilibrium state is detected by minimizing $\mathbf{u}^\top\mathbf{K}\mathbf{u}$ under constraint $\mathbf{u}^\top\mathbf{u} = 1$. The Lagrangian for this problem is written as

$$L_0(\mathbf{u}, \eta) = \mathbf{u}^\top\mathbf{K}\mathbf{u} + \eta(1 - \mathbf{u}^\top\mathbf{u}) \tag{22}$$

where η is the Lagrange multiplier. The stationary condition of L_0 with respect to $\mathbf{u}$ gives the eigenvalue problem

$$\mathbf{K}\mathbf{u} = \eta\mathbf{u} \tag{23}$$

for which η is regarded as the eigenvalue.

The Lagrangian for P5 is given as

$$L(\mathbf{t}, \eta) = \mathbf{t}^\top\mathbf{C}\mathbf{t} + \eta(1 - \mathbf{t}^\top\mathbf{t}) \tag{24}$$

Although μ_i in $\mathbf{C}$ is defined iteratively depending on the constraint activity in Step 4 of the inverse-power method, it is assumed here that the algorithm has been converged and the active constraints have been determined to fix the penalty parameters.

From the stationary conditions of L with (15), (19) and (20), we obtain

$$\mathbf{K}^+\mathbf{u} + \sum_{i=1}^{m} \mu_i\mathbf{h}_i(\mathbf{h}_i^\top\mathbf{u} - d_i^+) + \lambda^*\mathbf{u} = \eta\mathbf{u} \tag{25}$$

$$k_i d_i^+ - \mu_i(\mathbf{h}_i^\top\mathbf{u} - d_i^+) + \lambda^* d_i^+ = \eta d_i^+, \quad (i = 1, \dots, m) \tag{26}$$

Note that the constraint $\mathbf{h}_i^\top\mathbf{u} - d_i^+ \leq 0$ is not satisfied in exact equality in this penalty approach, and $\mu_i(\mathbf{h}_i^\top\mathbf{u} - d_i^+)$ in the second terms in (25) and (26) corresponds to the axial force due to elongation of a member in $\mathcal{I}$. We can also see from (23) and (25) that the eigenvalue is increased by λ^* due to the existence of the term $\lambda^*\mathbf{u}$ in the left-hand-side of (25).

Next we investigate the convergence properties with respect to the penalty parameter μ. The following equation is obtained from the optimality conditions of the original problem P4:

$$\mathbf{A}^*\mathbf{t}^0 - \eta^0\mathbf{t}^0 + \frac{1}{2}\sum_{j\in\mathcal{J}} \mu_j^0\mathbf{g}_j = \mathbf{0} \tag{27}$$

Table 1
Solutions of Ex1 for various values of penalty parameter.

μ	x_1	x_2	x_3	V	Δ	δ
1	1.8970×10^{-7}	0.52573	0.85065	1.2764	1.3131×10^{-1}	3.2492×10^{-1}
10	7.6515×10^{-7}	0.68923	0.72455	1.4750	2.4979×10^{-2}	3.5322×10^{-2}
100	8.8476×10^{-7}	0.70534	0.70887	1.4975	2.5000×10^{-3}	3.5355×10^{-3}
1000	8.9769×10^{-7}	0.70693	0.70728	1.4998	2.5017×10^{-4}	3.5355×10^{-3}

where $\mathbf{t}^0$ is the optimal value of $\mathbf{t}$, η^0 and μ_j^0 are the Lagrange multipliers, and $\mathcal{J} \subseteq \mathcal{I}$ is the set of indices of the active constraints at the optimal solution.

On the other hand, the solution $\mathbf{t}$ of P5 obtained by the inverse-power method satisfies

$$\left[\mathbf{A}^* + \mu \sum_{j \in \mathcal{J}} \mathbf{g}_j \mathbf{g}_j^\top\right]\mathbf{t} = \lambda \mathbf{t} \tag{28}$$

where μ is the specified penalty parameter, and λ is regarded as the eigenvalue of the matrix $[\mathbf{A}^* + \mu \sum_{j \in \mathcal{J}} \mathbf{g}_j \mathbf{g}_j^\top]$.

From (27) and (28), we obtain

$$\mathbf{A}^*(\mathbf{t}^0 - \mathbf{t}) + (\lambda \mathbf{t} - \eta^0 \mathbf{t}^0) + \sum_{j \in \mathcal{J}} (\mu_j^0/2 - \mu \mathbf{g}_j^\top \mathbf{t})\mathbf{g}_j = \mathbf{0} \tag{29}$$

Therefore, if $\mathbf{g}_j^\top \mathbf{t}$ converges to $\mu_i^0/(2\mu)$ as μ is increased, then $\mathbf{t}$ converges to $\mathbf{t}^0$ with $\lambda \to \eta^0$; i.e., if the error $\mathbf{g}_j^\top \mathbf{t}$ of an active constraint is inversely proportional to μ, then the error can be reduced to a small value by increasing μ. Note that too large value of μ results in illconditioning of the matrix $\mathbf{C}$.

6 Numerical examples

The convergence property of the algorithm proposed in Section 3 is first investigated by a small test problem which can be solved analytically. Stability of a small cable–bar structure is next investigated to confirm convergence to the optimal solution. Finally, a moderately large cable–bar structure is solved to ensure the practical applicability.

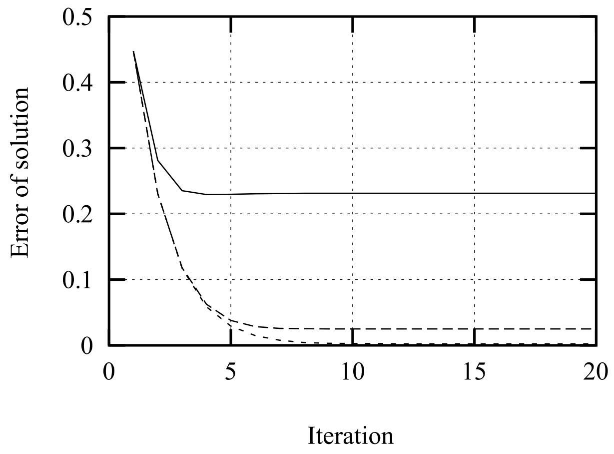
6.1 Small test problem

Consider first a small numerical example as

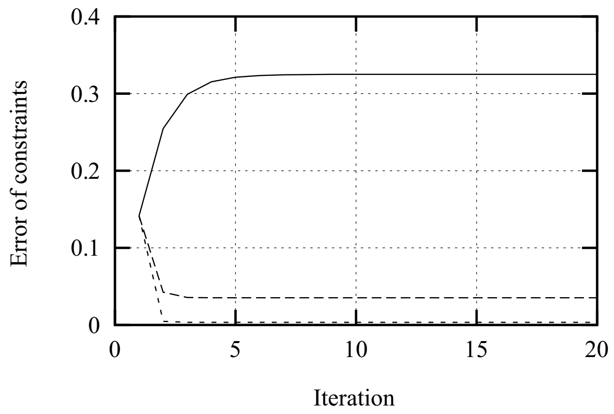
$$\begin{aligned} \text{Ex1}: \ \ & \text{minimize} \ \ V(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 && \text{(30a)} \\ & \text{subject to} \ \ x_2 \geq x_3 && \text{(30b)} \\ & \qquad\qquad\quad \mathbf{x}^\top \mathbf{x} = 1 && \text{(30c)} \end{aligned}$$

The optimal solution is easily found as $\mathbf{x}^{\text{opt}} = (0, 1/\sqrt{2}, 1/\sqrt{2})^\top$ with $V(\mathbf{x}^{\text{opt}}) = 1.5$, where the inequality constraint (30b) is active at the optimal solution.

The errors Δ and δ of the solution and the active constraint are defined as



(a) History of Δ.



(b) History of δ.

Fig. 1. Histories of the errors Δ and δ for Ex1; solid line: $\mu = 1$, dashed line: $\mu = 10$, dotted line: $\mu = 100$.

$$\Delta = \sqrt{\sum_{i=1}^{3}(x_i - x_i^{\text{opt}})^2} \tag{31a}$$

$$\delta = |x_2 - x_3| \tag{31b}$$

The results of 20 iterations from the initial solution $\mathbf{x} = (0.6, 0.8, 1.0)^\top$ with different values of penalty parameter are shown in Table 1. As is seen, the results strongly depend on the value of the penalty parameter. The histories of Δ and δ for $\mu = 1$, 10 and 100 are plotted in Figs. 1(a) and (b). For $\mu = 100$, the solution converges rapidly to a good approximate optimal solution with $\Delta = 2.5000 \times 10^{-3}$. The algorithm converged to the same value $\Delta = 2.5000 \times 10^{-3}$ in 20 steps for $\mu = 100$ from ten different randomly generated initial solutions. Therefore, the algorithm is robust in the sense that the solution does not depend on the initial value.

The values of Δ and δ at the 20th step are 2.5000×10^{-4} and 3.5355×10^{-4}, respectively, for $\mu = 1000$. Therefore, the errors are inversely proportional to the penalty parameter and converge to $\mu\Delta = 0.25000$, $\mu\delta = 0.35355$. The solution is not sensitive to μ if it is

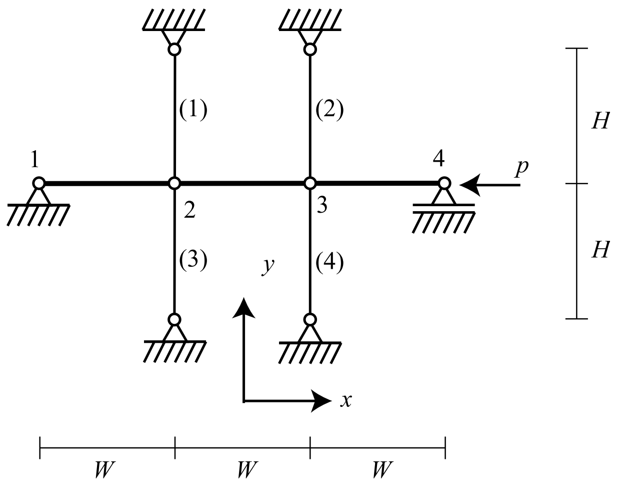


Fig. 2. Cable–bar Model 1.

moderately large; i.e., no trial-and-error process is needed for tuning the penalty parameter.

6.2 Cable–bar Model 1

Consider next a cable–bar Model 1 as shown in Fig. 2, where the horizontal bars are supported by the vertical cables. The bars and the cables in tensile state are modeled by the truss element. Let $H = W = 1$, and Young's modulus is 1, for simplicity. The cross-sectional areas are 100.0 for the bars and 1.0 for the cables. A horizontal load $p = 10.0$ is applied at support 4. All the cables have zero extension at the equilibrium state and are included in $\mathcal{I}$ in (2).

The minimum eigenvalue of $\mathbf{A}$ is $\lambda_1^A = -30.0$, and the maximum eigenvalue of the penalty matrix $\mathbf{P}$ is 3.0. It is known in the inverse-power method that a large ratio of the second eigenvalue to the lowest leads to rapid convergence to the lowest eigenvalue. Therefore, we define λ^* to be equal to $1.01|\lambda_1^A|$.

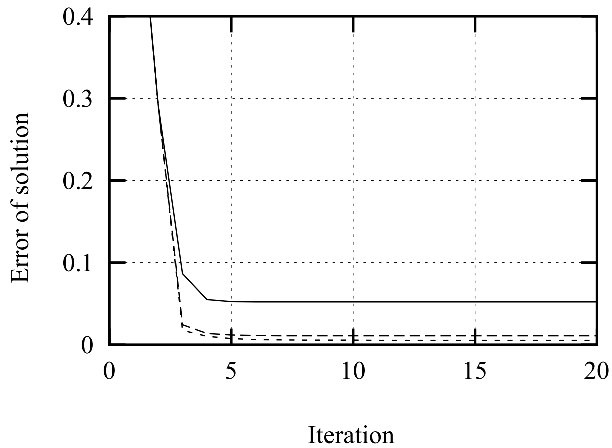
Let u_i $(i = 2, 3)$ denote the vertical incremental displacement of node i. The results of 20 iterations from a randomly generated initial solution with different values of penalty parameter are shown in Table 2. As is seen, the convergent solutions strongly depend on the value of the penalty parameter.

The optimal values u_i^{opt} of u_i are $(u_2^{\text{opt}}, u_3^{\text{opt}}) = (-0.5, 0.5)$ and the remaining displacement components are 0; i.e., the incremental displacements are antisymmetric with respect to the y-axis. The errors Δ and δ of the solution and the active constraints are defined as

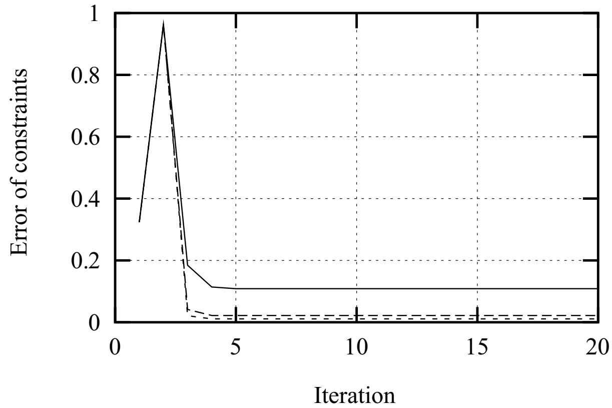
$$\Delta = \sqrt{\sum_{i=1}^{n} (u_i - u_i^{\text{opt}})^2} \tag{32a}$$

$$\delta = \sqrt{\sum_{j \in \mathcal{J}} (\mathbf{h}_i \mathbf{u} - d_i^{+})^2} \tag{32b}$$

The histories of Δ and δ for $\mu = 100$, 500 and 1000 are plotted in Fig. 3. For $\mu = 10000$, the



(a) History of Δ.



(b) History of δ.

Fig. 3. History of the error for Model 1; solid line: $\mu = 100$, dashed line: $\mu = 500$, dotted line: $\mu = 1000$.

solution converges rapidly to a good approximate optimal solution with $\Delta = 5.4781 \times 10^{-4}$. The errors are inversely proportional to the penalty parameter also for this case, and converge to $\mu\Delta = 5.4781$, $\mu\delta = 10.960$. Therefore, the solution is not sensitive to μ if it is moderately large; i.e., no trial-and-error process is needed for tuning the penalty parameter. The value of μ has been increased as $\mu = 10^6, 10^7, \dots$. The error decreases for $\mu \leq 10^{12}$ and increases as μ is further increased. The matrix $\mathbf{C}$ becomes singular at $\mu = 10^{17}$. However, a solution with sufficiently small error can be obtained for a wide range of μ.

The optimal value of V is 15.776, which is less than λ^*. Therefore, the equilibrium state is unstable. If we assume the symmetric displacement increment $(u_2, u_3) = (0.5, 0.5)$, where the remaining components are 0, the value of V is 28.800, which confirms that the displacement increment corresponding to the maximum decrease of the potential energy is antisymmetric with respect to the y-axis.

Table 2
Solutions of Model 1 for various values of penalty parameter.

μ	u_2	u_3	V	Δ	δ
100	-0.53693	0.53693	13.426	5.2226×10^{-2}	1.0863×10^{-1}
1000	-0.50386	0.50386	15.560	5.4585×10^{-3}	1.0959×10^{-2}
10000	-0.50039	0.50039	15.776	5.4781×10^{-4}	1.0960×10^{-3}

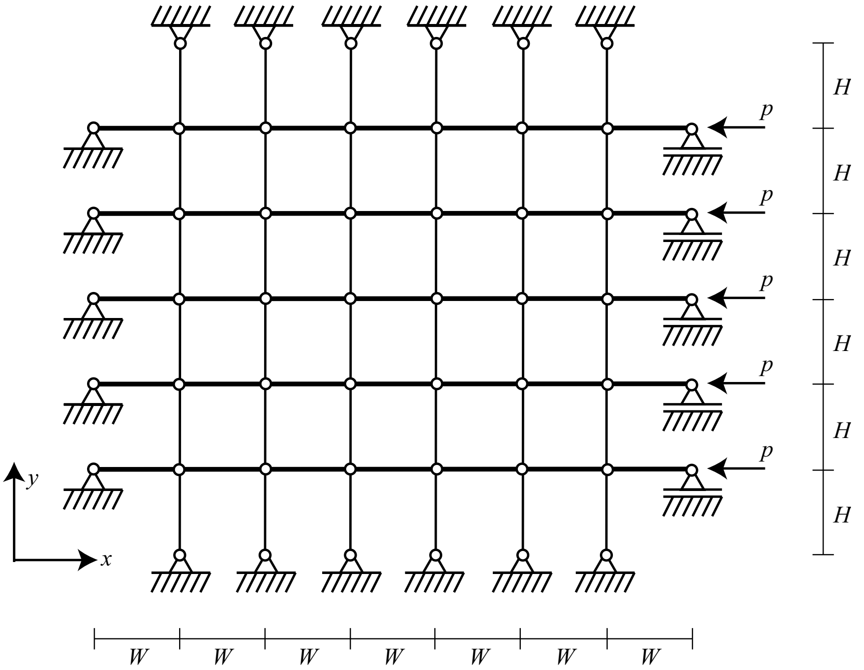


Fig. 4. Cable–bar Model 2.

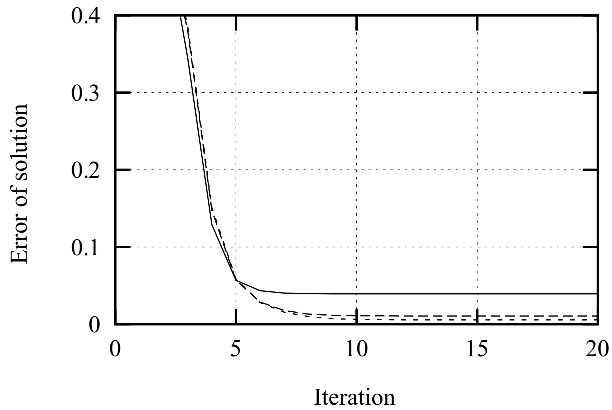
6.3 Cable–bar Model 2

Consider next a cable–bar model as shown in Fig. 4, where the horizontal bars are supported by the vertical cables. Let $H = W = 1$, and Young's modulus is 1, for simplicity. The cross-sectional areas are 100.0 for the bars and 1.0 for the cables. A horizontal load $p = 1.0$ is applied at each roller support. All the cables have zero extension at the equilibrium state and are included in $\mathcal{I}$ in (2).

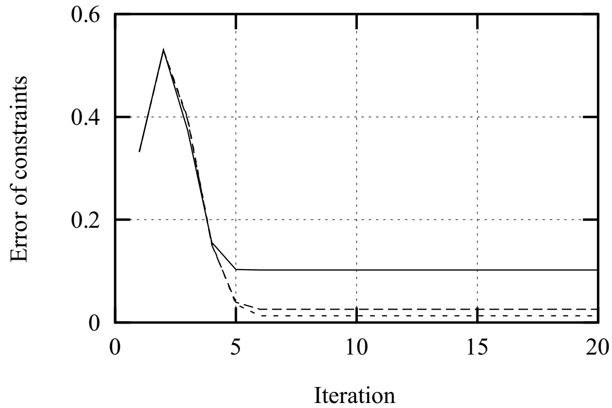
The minimum eigenvalue of $\mathbf{A}$ is $\lambda_1^A = -3.8478$. Since the ratio of the second eigenvalue to the lowest of $\mathbf{A}$ should be large enough, we define λ^* to be equal to $1.01|\lambda_1^A|$ also for this example.

The solution for $\mu = 10^6$ is regarded as the optimal solution. The errors in (32) are also used in this example. The histories of the errors Δ and δ are plotted for $\mu = 10$, 50 and 100 in Figs. 5(a) and (b). The solution converges rapidly to a good approximate optimal solution with $\Delta = 5.6088 \times 10^{-3}$ if we choose $\mu = 100$. The optimal incremental displacement for $\mu = 10^6$ is plotted in Fig. 6.

Although $\mu\Delta$ did not converge in this example due to numerical oscillation, the error δ of constraints is inversely proportional to μ, and $\mu\delta$ converged to 1.3696. The optimal value of V is 0.56180, which is less than λ^*. Therefore, the equilibrium state is unstable.



(a) History of Δ.



(b) History of δ.

Fig. 5. History of the error for Model 2; solid line: $\mu = 10$, dashed line: $\mu = 50$, dotted line: $\mu = 100$.

7 Conclusions

The stability analysis problem of elastic conservative systems with discontinuity in extensional stiffness of a cable has been formulated as a minimization problem of convex quadratic function under linear inequality constraints and a single quadratic equality constraint. The problem is solved by an iterative algorithm based on the inverse-power method for eigenvalue analysis. The conclusions obtained from this study are summarized as follows:

(1) Instability of an equilibrium state of a cable–bar structure can be detected by solving a minimization problem of the incremental total potential energy over the compatibility conditions.

(2) The non-convex incremental total potential energy for an unstable state can be converted to a convex quadratic function by using a shifting operator. The discontinuity in extensional stiffness of a cable can be incorporated as a convex penalty term using the slack variables.

(3) The minimization problem of the convex quadratic function under linear inequality constraints and a quadratic equality constraint can be solved by the inverse-power method. This way, the difficulty due to nonconvexity of the potential energy at an

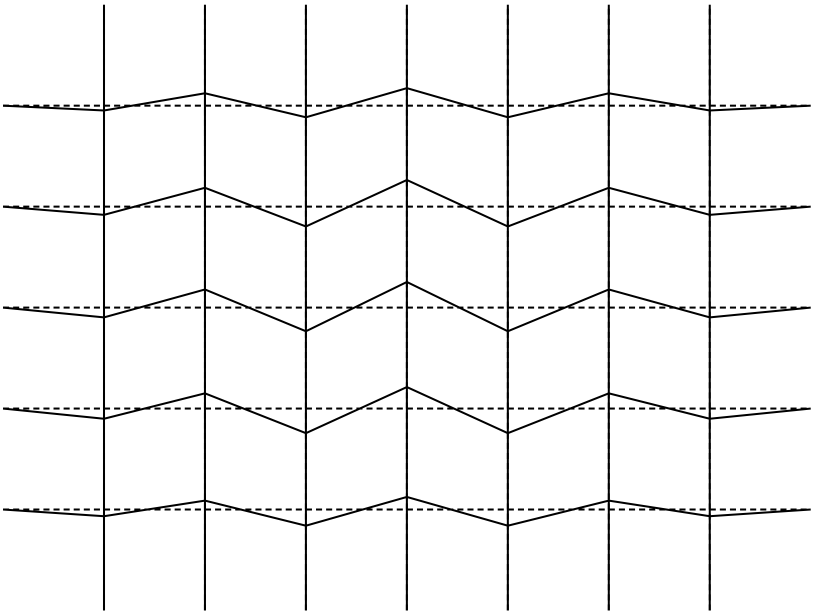


Fig. 6. Incremental displacement corresponding to the minimum of the incremental potential energy

for Model 2.

unstable equilibrium state has been successfully overcome.

(4) The error of the active constraint is inversely proportional to the penalty parameter. Therefore, the error can be reduced to an arbitrary small value by increasing the penalty parameter.

(5) The numerical examples show that the iterative process converges in about ten steps irrespective of the size of the structure. Another advantage of the method is that the solution converges to the exact value as the penalty parameter is increased. Therefore, a moderate value of the penalty parameter can be assigned with a few trial steps.

References

Atkinson, K. E. (1989). *An Introduction to Numerical Analysis*. John Wiley & Sons, 2nd edition.

Bigoni, D. (2000). Bifurcation and instability of non-associative elastoplastic solids. In H. Petryk, editor, *Material Instabilities in Elastic and Plastic Solids*, pages 1–52. Springer.

Choong, K. K. and Hangai, Y. (1993). Bifurcation analysis of a link model supported by nonequi-resistant boundary springs. In *Proc. Seiken-IASS Symposium on Nonlinear Analysis and Design for Shell and Spatial Structures*, Tokyo.

Costa, A. P. d., Martins, J. A. C., Figueiredo, I. N., and Júdice, J. J. (2004). The directional instability problem in systems with frictional contacts. *Comp. Meth. Appl. Mech. Engng.*, **193**, 357–384.

Hill, R. (1958). A general theory of uniqueness and stability in elastic-plastic solids. *J.*

Mech. Phys. Solids, **6**, 236–249.

Kanno, Y. and Ohsaki, M. (2003). Minimum principle of complementary energy of cable networks by using second-order cone programming. *Int. J. Solids Struct.*, **40**(17), 4437–4460.

Ohsaki, M. and Zhang, J. Y. (2006). Stability conditions of prestressed pin-jointed structures. *Int. J. Non-Linear Mech.*, **41**, 1109–1117.

Panagiotopoulos, P. D. (1976). A variational inequality approach to the inelastic stress-unilateral analysis of cable-structures. *Comp. & Struct.*, **6**, 133–139.

Pignataro, M., Rizzi, N., and Luongo, A. (1991). *Stability, Bifurcation and Postcritical Behaviour of Elastic Structures*, volume 39 of *Dev. Civil Eng.* Elsevier, Amsterdam.

Riks, E. (1998). Buckling analysis of elastic structures: a computational approach. In *Advances in Applied Mechanics*, volume 34, pages 1–76. Academic Press, San Diego, CA.

Salle, J. L. and Lefscetz, S. (1961). *Stability by Liapunov's Direct Method.* Academic Press, New York.

Thompson, J. M. T. and Hunt, G. W. (1973). *A General Theory of Elastic Stability.* John Wiley, New York, NY.

Tschöpe, H., Oñate, E., and Wriggers, P. (2003). Direct computation of instability points for contact problem. *Comp. Mech.*, **31**, 173–178.

Villagio, P. (1979). Buckling under unilateral constraints. *Int. J. Solids Struct.*, **15**, 193–201.

Wriggers, P. and Simo, J. C. (1990). A general procedure for the direct computation of turning and bifurcation points. *Int. J. Num. Meth. Engng.*, **30**, 155–176.

Zhang, J. Y. and Ohsaki, M. (2007). Stability conditions for tensegrity structures. *Int. J. Solids Struct.*, **44**(11-12), 3875–3886.