Local and Global Searches of Approximate Optimal Designs of Regular Frames[1]

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Abstract

Methods of local and global searches of approximate optimal designs minimizing total structural volume under stress and displacement constraints are presented for regular frames subjected to static loads. Nonuniqueness of the optimal solution is extensively utilized for local search of approximate optimal solutions, where the search direction is computed from singular value decomposition of the stiffness matrix with respect to the cross-sectional areas, or the sensitivity matrix of the constraints. The distance between the solutions is then defined, and the approximate optimal solutions are globally and consecutively found so as to maximize the distance from the already found solutions under upper bound constraint on the total structural volume. The effectiveness of the proposed method is demonstrated in application to a plane frame.

Keywords Optimum design; Singular value decomposition; Regular frame; Stress constraints; Nonuniqueness of optimal solution

1 Introduction

Optimization of frames to minimize the total structural volume under stress and displacement constraints for static loads is a standard problem of structural optimization [1], and numerous number of results and solution methodologies have been presented [2]. Optimization under stress constraints has traditionally been solved by fully stressed design (FSD), which is first presented by Michell [3]. Although optimization problem under stress constraints is not equivalent to the FSD even for simple trusses if the solution is statically indeterminate [4, 5, 6], the FSD is practically acceptable because it generates approximate optimal solutions with small computational cost.

Recently, it has been pointed out that there may exist many FSDs with almost the same total structural volume [7, 8]. Therefore, obtaining only one solution will not be enough for practical purpose, where several solutions satisfying stress constraints should be compared in view of other performance measures such as eigenfrequencies and requirements in construction process. Furthermore, the objective function may not be strictly minimized; i.e., it will be helpful for the designers if several approximate solutions with different distributions of cross-sectional areas are obtained.

Nonuniqueness of the optimal solutions can be classified as:

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(a) Local nonuniqueness due to independence of the displacements and the constraint functions on the design variables, which is enhanced by regularity of the frame.
(b) Global nonuniqueness due to nonconvexity of the objective and/or constraint functions.

Similar nonunique property can be observed for a plate discretized to finite elements [9]. The nonuniqueness of the solution has been thought to have negative effect which deteriorates the convergence property of the optimization algorithm. Jog and Haber [10] suggested that the nonuniqueness of the solution to a compliance optimization problem can be detected by the singular values [11] of the matrix defined as the gradients of the equivalent force vector with respect to the design variables. However, they did not show how the singular vectors are used for finding approximate optimal solutions. Furthermore, there is no general approach to optimization problems with; e.g., stress and displacement constraints.

The nonconvex and nonunique properties, however, can be extensively used to find many approximate optimal solutions. In this paper, a local search method is first presented based on Singular Value Decomposition (SVD) of the stiffness matrix with respect to the cross-sectional areas and the sensitivity matrix of the constraints of the framed structures. The distance between the solutions is then defined, and the approximate optimal solutions are consecutively found so as to maximize the distance from the already found approximate solutions under constraint on upper bound of the total structural volume. The effectiveness of the proposed methods is demonstrated in application to a 6×6 plane frame.

2 Optimization problem

Consider a rigidly-jointed regular plane frame as shown in Fig. 1. Let A_i^* denote the cross-sectional area of the ith member. The second moment of area I_i^* and the section

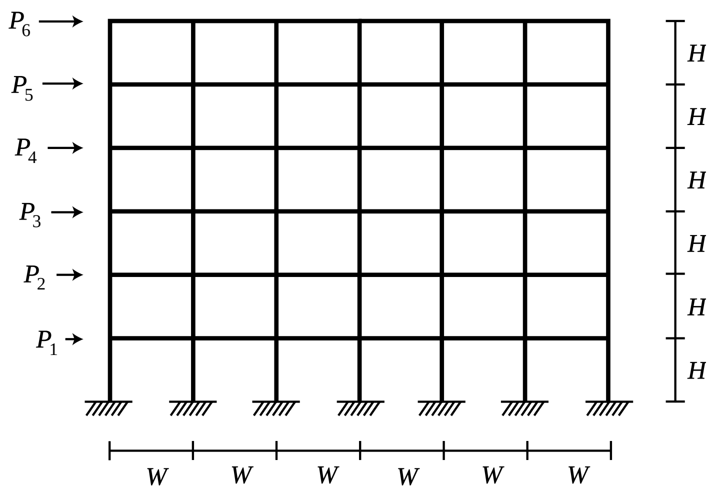


Fig. 1: A 6-span 6-story frame.

modulus Z_i^* are defined as functions of A_i^*. Hence, A_i^* can be considered as the design variables for the following optimization problem.

The members are divided into groups based on the symmetry property, and the number of variables is reduced by using the design variable linking approach. Let A_i denote the cross-sectional area of the members in the ith group, and the vector consisting of A_i is denoted by $\mathbf{A}$. In the following, a component of a vector is indicated by a subscript.

Let $\mathbf{P}$ denote the static load vector. The nodal displacement vector against $\mathbf{P}$ is denoted by $\mathbf{U}$ that is found by solving the equilibrium equation as

$$\mathbf{K}(\mathbf{A})\mathbf{U} = \mathbf{P} \tag{1}$$

where $\mathbf{K}(\mathbf{A})$ is the stiffness matrix, and the deformation is assumed to be sufficiently small. The equivalent nodal loads $\mathbf{F}(\mathbf{U}, \mathbf{A})$ corresponding to $\mathbf{U}$ is defined by

$$\mathbf{F}(\mathbf{U}, \mathbf{A}) = \mathbf{K}(\mathbf{A})\mathbf{U} \tag{2}$$

The conventional assumption of rigid floor is used; i.e., the horizontal displacements of the nodes in the same story are same. Hence, there is no axial deformation in the beams. The stress constraints are assigned at the two ends of the members, and the same upper bound $\bar{\sigma}$ is given for the absolute values of the stresses. Therefore, we have four points in a column, i.e., two flanges of two ends, for which the stress should be constrained. However, the number of points for stress constraints is two for a beam, because the beam has no axial deformation. Constraints are also given for the nodal displacements.

Let $g_j(\mathbf{U}(\mathbf{A}), \mathbf{A})$ denote the jth constraint function of the optimization problem representing the bound on a response such as stress and displacement. The total length of the members in the ith group and the number of groups are denoted by L_i and m, respectively. The objective function is the total structural volume V. The following optimization problem is to be solved:

$$\text{OPT1}: \quad \text{minimize} \quad V(\mathbf{A}) = \sum_{i=1}^{m} A_i L_i \tag{3}$$

$$\text{subject to} \quad g_j(\mathbf{U}(\mathbf{A}), \mathbf{A}) \leq 0, \quad (j = 1, \dots, s) \tag{4}$$

$$A_i^{\mathrm{L}} \leq A_i \leq A_i^{\mathrm{U}}, \quad (i = 1, \dots, m) \tag{5}$$

where A_i^{L} and A_i^{U} are the lower and upper bounds for A_i, respectively, and s is the number of constraints. OPT1 is a nonlinear programming problem that can be solved by any optimization algorithm such as sequential quadratic programming.

3 Local Search of Approximate Optimal Solutions

Since OPT1 is nonconvex, convergence to the global optimal solution is not guaranteed. However, if the solutions with same objective value are always found starting from different initial solutions, the solutions can be conceived as globally optimal. Even if a solution is locally optimal, it satisfies the Karush–Kuhn–Tucker (KKT) conditions that are the necessary and sufficient conditions for local optimality [12].

Let μ_j denote the Lagrange multiplier for the jth constraint. The derivative S_i of the Lagrangian with respect to A_i is given as

$$S_i = L_i + \sum_{j=1}^{s} \mu_j \frac{\partial g_j}{\partial A_i}, \quad (i = 1, \dots, m) \tag{6}$$

The KKT conditions are written as

$$\begin{cases} S_i = 0 \;\text{ for }\; A_i^{\mathrm{L}} < A_i < A_i^{\mathrm{U}} \\ S_i \geq 0 \;\text{ for }\; A_i = A_i^{\mathrm{L}}, \\ S_i \leq 0 \;\text{ for }\; A_i = A_i^{\mathrm{U}} \end{cases} \qquad (i = 1, \dots, m) \tag{7}$$

with complementarity conditions

$$\mu_j \geq 0, \quad g_j \leq 0, \quad \mu_j g_j = 0, \quad (j = 1, \dots, s) \tag{8}$$

If $\mathbf{A}$ is locally optimal, there exist $\mathbf{U}$ and μ_j that satisfy the KKT conditions (7) and (8).

Let J^{E} denote the set of member groups that satisfy $A_i^{\mathrm{L}} < A_i < A_i^{\mathrm{U}}$. Eq. (7) for $i \in J^{\mathrm{E}}$ can be written as

$$L_i + \sum_{j=1}^{s} \mu_j \frac{\partial g_j}{\partial A_i} = 0, \quad (i \in J^{\mathrm{E}}) \tag{9}$$

The number of groups in J^{E} is denoted by n^{E}. Let C^{E} and s^{E} denote the set of independent active constraints and the number of constraints in C^{E}. For a solution with $s^{\mathrm{E}} \geq n^{\mathrm{E}}$, e.g., a FSD, μ_j can be computed from (9) after obtaining $\mathbf{A}$ and $\mathbf{U}$ by the active constraints $g_j = 0$ $(j \in C^{\mathrm{E}})$.

Suppose that g_j is a function of $\mathbf{U}$ only and does not explicitly depend on $\mathbf{A}$. If $\mathbf{A}$, $\mathbf{U}$ and μ_j satisfying the KKT conditions are obtained and there exist a direction vector $\Delta\mathbf{A}$ of $\mathbf{A}$ that does not have any effect on $\mathbf{U}$, then the optimal solution is locally nonunique within first order approximation. For the case where $\mathbf{P}$ is independent of $\mathbf{A}$, the incremental form of (1) for variation of $\mathbf{A}$ is written as

$$\mathbf{K}(\mathbf{A})\Delta\mathbf{U} + \mathbf{K}^{\mathrm{A}}(\mathbf{U},\mathbf{A})\Delta\mathbf{A} = \mathbf{0} \tag{10}$$

where $\mathbf{K}^{\mathrm{A}}(\mathbf{U},\mathbf{A})$ is given using (2) as

$$\mathbf{K}^{\mathrm{A}}(\mathbf{U},\mathbf{A}) = \frac{\partial \mathbf{F}(\mathbf{U},\mathbf{A})}{\partial \mathbf{A}} \tag{11}$$

which is called stiffness matrix with respect to the cross-sectional areas. The nonuniqueness of the optimal solution can be detected from (10) by computing the rank of $\mathbf{K}^{\mathrm{A}}$ that depends on $\mathbf{U}$.

For a frame with sandwich cross-section, where I_i^* is proportional to A_i^*, $\mathbf{K}^{\mathrm{A}}$ does not explicitly depend on $\mathbf{A}$, and the ith column is equal to the sum of the nodal load vectors corresponding to the unit value of A_i of the members in the ith group.

Let λ_i $(\lambda_1 \geq \lambda_2 \geq \dots)$ denote the ith singular value of $\mathbf{K}^{\mathrm{A}}$, and the ith diagonal component of the diagonal matrix $\mathbf{\Lambda}$ is λ_i. SVD of $\mathbf{K}^{\mathrm{A}}$ leads to

$$\mathbf{K}^{\mathrm{A}} = \mathbf{Q}\mathbf{\Lambda}\mathbf{R}^{\top} \tag{12}$$

where the ith columns of $\mathbf{Q}$ and $\mathbf{R}$ are the singular vectors $\mathbf{Q}_i$ and $\mathbf{R}_i$, respectively, corresponding to λ_i, and the following relation is satisfied:

$$\mathbf{K}^{\mathrm{A}}\mathbf{R}_i = \lambda_i \mathbf{Q}_i, \quad (i = 1, 2, \dots, m) \tag{13}$$

It is assumed here that m is less than the number of degrees of freedom n, which is usually satisfied by a rigidly-jointed regular frame. If $\mathbf{K}^{\mathrm{A}}$ is full rank, it can be observed from (10) that there is no solution $\Delta\mathbf{A}$ ($\neq \mathbf{0}$ satisfying $\Delta\mathbf{U} = \mathbf{0}$. For a non-optimal general frame, the rank of $\mathbf{K}^{\mathrm{A}}$ is m, because variation of $\mathbf{A}$ in any pattern will lead to variation of $\mathbf{U}$. However, it often happens for a regular frame as shown in the following examples that rank($\mathbf{K}^{\mathrm{A}}$) is less than m. Consider the case, e.g., where rank($\mathbf{K}^{\mathrm{A}}$) $= m - 1$. Since $\lambda_m = 0$,

$$\mathbf{K}^{\mathrm{A}}\mathbf{R}_m = \mathbf{0} \tag{14}$$

is satisfied. Therefore, $\Delta\mathbf{U} = \mathbf{0}$ from (10) for $\Delta\mathbf{A} = \mathbf{R}_m$ and $\mathbf{U}$ is constant within first-order approximation if $\mathbf{A}$ is modified in the direction of $\mathbf{R}_m$. The KKT conditions are satisfied by the modified design if n^{E} is not less than m, because μ_j can be found from (9). This way, optimal solutions can be locally searched using SVD of $\mathbf{K}^{\mathrm{A}}$.

Even if λ_m is not equal to 0 and has very small positive value compared to the maximum value λ_1, approximate optimal solutions can be found by searching in the direction of $\mathbf{R}_m$. However, in this case, more accurate directions can be found by directly using the SVD of the sensitivity matrix $\mathbf{G}$ for which the (j, i)-component is equal to the sensitivity coefficient of the jth constraint function with respect to the ith design variable as

$$\mathbf{G} = \left[\frac{\partial g_j}{\partial A_i}\right] \tag{15}$$

Note that the side constraints should be satisfied by the modified solution; i.e., $\Delta A_i \geq 0$ for groups in J^{L} satisfying $A_i = A_i^{\mathrm{L}}$, and $\Delta A_i \leq 0$ for groups in J^{U} satisfying $A_i = A_i^{\mathrm{U}}$. If it is not straightforward to find such feasible direction, it can be found by solving an auxiliary optimization problem as a linear combination of the singular vectors. Let e_i denote the coefficient for $\mathbf{R}_i = \{R_{i,j}\}$. The number of singular vectors to be considered is denoted by h. The coefficients are found by solving the following optimization problem to minimize an objective function $Q(\mathbf{e})$:

$$\text{OPT2}: \quad \text{minimize} \quad Q(\mathbf{e}) \tag{16}$$

$$\text{subject to} \quad \sum_{i=m-h+1}^{m} e_i R_{i,j} \geq 0, \quad (j \in J^{\mathrm{L}}) \tag{17}$$

$$\sum_{i=m-h+1}^{m} e_i R_{i,j} \leq 0, \quad (j \in J^{\mathrm{U}}) \tag{18}$$

$$\sum_{i=m-h+1}^{m} (e_i)^2 = 1 \tag{19}$$

Note that h is taken as the minimum value that has a feasible solution of OPT2. Since the purpose of solving OPT2 is to find a direction satisfying the constraints (17) and (18), $Q(\mathbf{e})$ can be given arbitrary. Another norm constraint can also be given for (19).

4 Global Search of Approximate Optimal Solutions

Since OPT1 is a nonconvex problem, there may exist many local optimal solutions that have slightly larger objective values than that of the global optimal solution. Also, for a regular frame, we can find approximate optimal solutions in the neighborhood of a local optimal solution as described in the previous section. In view of practical application, the most preferred solution should be chosen from a set of approximate solutions considering other performance criteria.

Let $\tilde{V}$ denote the optimal objective value of OPT1. Assign the following requirement as an approximate optimal solution:

$$V(\mathbf{A}) \leq \tilde{V} + \Delta V \tag{20}$$

where ΔV is assumed to be sufficiently small. Let $\tilde{\mathbf{A}}^k$ denote the kth approximate optimal solution that has been already found. The distance D^k between $\mathbf{A}$ and $\tilde{\mathbf{A}}^k$ is defined by the Euclidean norm as

$$D^k = \sqrt{\sum_{i=1}^{m} (A_i - \tilde{A}_i^k)^2} \tag{21}$$

Let τ be an auxiliary variable, and the following optimization problem is solved for maximizing the minimum distance from existing optimal solutions under constraints on the responses and the total structural volume:

$$\begin{aligned} \text{OPT3}: \ \text{maximize} \quad & \tau \\ \text{subject to} \quad & \tau \leq \sqrt{\sum_{i=1}^{m}(A_i - \tilde{A}_i^k)^2}, \quad (k = 1, 2, \ldots, q) & (22) \\ & V(\mathbf{A}) \leq \tilde{V} + \Delta V & (23) \\ & g_j(\mathbf{U}(\mathbf{A}), \mathbf{A}) \leq 0, \quad (j = 1, \ldots, s) & (24) \\ & A_i^{\mathrm{L}} \leq A_i \leq A_i^{\mathrm{U}}, \quad (i = 1, \ldots, m) & (25) \end{aligned}$$

where q is the number of approximate solutions that have been already found. This way, approximate optimal solutions with various distributions of cross-sectional areas can be found consecutively by solving OPT3.

5 An Illustrative Example

In the following examples, including the plane frame in the next section, the units of force and length are kN and mm, respectively. Optimization is carried out by IDESIGN Ver. 3.5 [13], where the sequential quadratic programming is used. Sensitivity coefficients of stress and displacement are found using the standard approach of direct differentiation method.

Nonuniqueness of the optimal solution is first investigated by a continuous beam as shown in Fig. 2, where the numbers with and without parentheses are member numbers and node numbers, respectively. The length of each beam is 2000, and the elastic modulus is 200. The lower bound A_i^{L} for the cross-sectional area is 100. Upper bound is not given for A_i. The applied nodal moment is 10000. Design variable linking is not used, i.e., the number of variables m is 6.

Consider a sandwich cross-section defined by

$$I_i^* = (r_i^*)^2 A_i^* \tag{26}$$

where $r_i^* = 50$ is the radius of the ith member. Constraints are given for the stresses at two ends of each member.

Let θ_i denote the rotation of the ith node. To investigate the special case of nonunique optimal solution, we assign a periodic boundary condition such that $\theta_1 = \theta_7$. Uniform random number $t_i \in [0, 1)$ is generated to define the initial solution as

$$A_i = A_0(1 + ct_i), \quad (i = 1, \ldots, m) \tag{27}$$

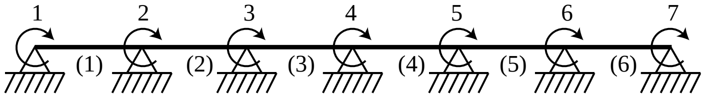


Fig. 2: A 6-span continuous beam.

Table 1: Optimization results of the beam.

	$c = 0$	$c = 0.5$	$c = 1.0$	$c = 2.0$
A_1	714.286	717.170	719.964	724.229
A_2	714.286	711.401	708.607	704.343
V ($\times 10^6$)	8.57143	8.57143	8.57143	8.57143

where c is a parameter, and $A_0 = 100$ for this example.

Table 1 shows the optimization results from various initial solutions. Note that each optimal solution has been found with 8 iterations, and satisfies at least one of the stress constraints in each member in equality. All the solutions are periodic such that $A_1 = A_3 = A_5$ and $A_2 = A_4 = A_6$. It is seen from Table 1 that the objective value is same for all cases, although the cross-sectional areas are different. The nodal rotations have the same value 4.6667×10^{-3}.

The rank of $\mathbf{K}^{\mathrm{A}}$ is 5, and the singular values are 14.000, 12.124, 12.124, 7.0000, 7.0000, 0.0. The singular vector $\mathbf{R}_6$ corresponding to the vanishing singular value is $(-1, 1, -1, 1, -1, 1)$ after normalization. Therefore, the optimal solutions can be written with a parameter α as

$$\begin{aligned} A_1 = A_3 = A_5 = 714.286 + \alpha, \\ A_2 = A_4 = A_6 = 714.286 - \alpha \end{aligned} \tag{28}$$

which agree with the results in Table 1.

If the constraint $\theta_1 = \theta_7$ of the periodic boundary condition is not given, the rank of $\mathbf{K}^{\mathrm{A}}$ of the optimal solution is 6, and the singular values are 12.064, 11.151, 9.1058, 8.3791, 5.3656, 2.7534. Therefore, the rank deficiency of $\mathbf{K}^{\mathrm{A}}$ is related to the regularity of the frame.

6 Numerical Example of a Regular Frame

6.1 Description of the frame model

Optimal solutions are found for a 6-story 6-span frame as shown in Fig. 1, where $H = W = 4000$, $m^* = 84$, $n = 97$, and $m = 45$ considering the symmetry condition. Note that the units of force and length are kN and mm also in this section. Only horizontal loads are applied, and $(P_1, P_2, P_3, P_4, P_5, P_6) = (50, 100, 150, 200, 250, 300)$ in Fig. 1. The elastic modulus is 200. The upper bound A_i^{U} is not given for A_i. The objective function in OPT2 is given as

$$Q(\mathbf{e}) = \sum_{i=1}^{h} \sum_{j=1}^{m} e_i R_{i,j} \tag{29}$$

Table 2: Optimization results of the frame.

	$c = 0$	$c = 0.5$	$c = 1.0$	$c = 2.0$
A_1	22047.3	22041.2	22040.5	22036.4
A_2	14026.0	13992.8	14008.7	13983.8
A_3	8266.1	8265.3	8274.1	8271.7
A_4	4474.1	4486.2	4473.1	4484.5
V $(\times 10^9)$	2.4824	2.4824	2.4824	2.4824

6.2 Local search for frame with sandwich section

The lower bound for the cross-sectional area is 3000, and $r_i^* = 250$ for all members. The initial solutions are randomly generated by (27) with $A_0 = 3000$.

The optimal objective values for all the cases are 2.4824×10^9, although the cross-sectional areas are different as shown in Table 2. The values of $A_1, \dots, A_4$ of the external column of stories 1–4 are also shown in Table 2. It is observed that the optimization algorithm converges to global optimal solutions, but the uniqueness of the solution is not satisfied. The optimal solution for $c = 0$, denoted by optimal solution 1, is shown in Fig. 3, where the width of each member is proportional to its cross-sectional area.

The optimal solution is fully stressed; i.e., the maximum absolute values of the stresses are equal to the upper bound $\bar{\sigma}$ for all groups. The number of active constraints at the optimal solution is 126, which is larger than n $(= 97)$, for all the cases. Therefore, the displacements are uniquely determined by the active constraints.

SVD has been carried out for $\mathbf{K}^{\mathrm{A}}$ of the optimal solution 1 in Fig. 3. The 20 lowest and the maximum singular values are listed in the 1st column of Table 3. Note that $\lambda_{45}, \dots, \lambda_{29}$ are relatively small compared with the maximum value λ_1, but λ_{28} is about 8 times as large as λ_{29}.

Let σ_{ij} denote the stress at the jth point where the stress is constrained in the ith member. The accuracy of an approximate solution is confirmed by the maximum stress ratio β defined by

$$\beta = \max_{i,j} \left(\frac{|\sigma_{ij}|}{\bar{\sigma}} \right) \tag{30}$$

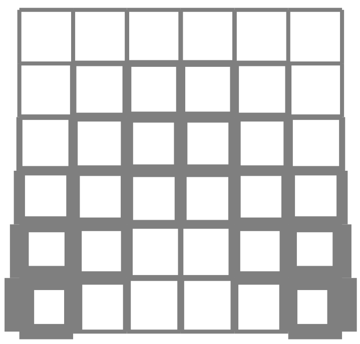


Fig. 3: Optimal solution 1 (sandwich section)

Table 3: Singular values of the optimal solutions with sandwich section.

	Sandwich section		H-beam and box-solumn	
	$\mathbf{K}^{\mathrm{A}}$	$\mathbf{G}$	$\mathbf{K}^{\mathrm{A}}$	$\mathbf{G}$
λ_{45}	5.9151×10^{-3}	1.0672×10^{-20}	1.0114×10^{-2}	4.0244×10^{-20}
λ_{44}	8.3172×10^{-3}	1.1750×10^{-20}	1.1006×10^{-2}	5.5420×10^{-20}
λ_{43}	1.3406×10^{-2}	1.5271×10^{-20}	1.1598×10^{-2}	6.6670×10^{-20}
λ_{42}	1.7397×10^{-2}	2.3706×10^{-20}	1.5443×10^{-2}	8.4620×10^{-20}
λ_{41}	1.9328×10^{-2}	3.0968×10^{-20}	1.7206×10^{-2}	1.1879×10^{-19}
λ_{40}	2.0601×10^{-2}	3.8565×10^{-20}	2.2926×10^{-2}	1.5696×10^{-19}
λ_{39}	2.4255×10^{-2}	4.5570×10^{-20}	2.3577×10^{-2}	1.6586×10^{-19}
λ_{38}	2.7042×10^{-2}	4.7258×10^{-20}	2.5175×10^{-2}	3.3047×10^{-19}
λ_{37}	2.9890×10^{-2}	5.4876×10^{-20}	3.3748×10^{-2}	4.7740×10^{-19}
λ_{36}	3.6977×10^{-2}	5.8879×10^{-20}	3.7028×10^{-2}	9.3395×10^{-19}
λ_{35}	4.0571×10^{-2}	9.1018×10^{-20}	4.0585×10^{-2}	2.9089×10^{-18}
λ_{34}	4.2824×10^{-2}	9.7051×10^{-20}	4.5260×10^{-2}	7.4879×10^{-6}
λ_{33}	5.2246×10^{-2}	1.0483×10^{-19}	5.3971×10^{-2}	1.1227×10^{-5}
λ_{32}	5.8112×10^{-2}	1.1878×10^{-19}	6.4710×10^{-2}	1.3807×10^{-5}
λ_{31}	6.7549×10^{-2}	1.5394×10^{-19}	7.6041×10^{-2}	1.5792×10^{-5}
λ_{30}	8.3303×10^{-2}	2.1828×10^{-19}	7.8555×10^{-2}	1.8081×10^{-5}
λ_{29}	1.0452×10^{-1}	2.5267×10^{-19}	9.7808×10^{-2}	3.0958×10^{-5}
λ_{28}	7.8204×10^{-1}	4.4907×10^{-6}	3.2119	3.4460×10^{-5}
λ_{27}	18.313	1.7936×10^{-5}	15.193	4.3629×10^{-5}
λ_{26}	33.031	2.5408×10^{-5}	24.993	4.9311×10^{-5}
...	...	...	...	...
λ_1	124.528	3.38253×10^{-4}	177.304	4.46112×10^{-4}

where $\beta = 1$ for the optimal solution. Since all the components $R_{1,i}$ $(i \in J^{\mathrm{L}})$ have positive values for this example, the cross-sectional areas are parametrically varied in the direction of the singular vector $\mathbf{R}_{45}$ corresponding to the lowest singular value as $\mathbf{A} + \alpha\mathbf{R}_{45}$, where α is a parameter. The maximum value of $|R_{1,i}|/A_i$ among all the groups is denoted by η. The ratio of V to that of the optimal solution is denoted by γ. Variations of β and γ with respect to the cross-sectional parameter η are as shown in Fig. 4.

For example, if $\eta = 0.1$; i.e., if we allow 10% variation of A_i, then $\beta = 1.0024$, and the increase of the maximum stress ratio β from 1 is very small compared with η. The ratio γ of the total structural volume V for $\eta = 0.1$ is 0.99898, i.e., the variation of the objective value is very small, but it decreases in the process of searching the approximate solutions.

Next, SVD has been carried out for $\mathbf{G}$ of the optimal solution 1 in Fig. 3. The 20 lowest and the maximum singular values are listed in the 2nd column of Table 3. Note that the singular values for $\mathbf{G}$ are much smaller than those of $\mathbf{K}^{\mathrm{A}}$. However, the ratio of the minimum value λ_{45} to the maximum value λ_1 plays the crucial role for accuracy of the approximate solutions. Since λ_{45}/λ_1 has smaller value in $\mathbf{G}$ than in $\mathbf{K}^{\mathrm{A}}$, more accurate solutions may be expected to be obtained by the singular vectors of $\mathbf{G}$ than those of $\mathbf{K}^{\mathrm{A}}$.

The problem OPT2 has been solved to obtain the feasible direction satisfying the side constraints. The vector $\mathbf{R}_i$ is normalized by $\mathbf{R}_i^\top \mathbf{R}_i = 1$. The value of h is 4 for

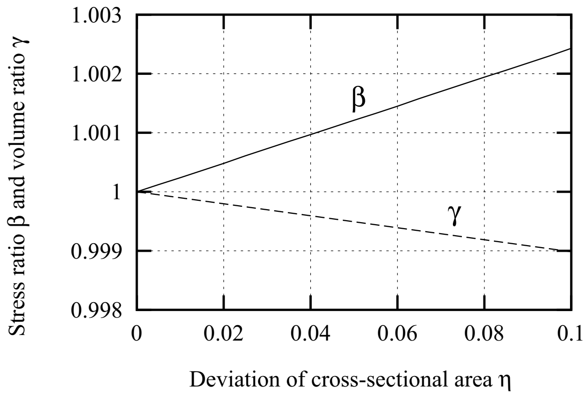


Fig. 4: Variation of the maximum stress ratio β (solid line) and the volume ratio γ (dashed line) with respect to cross-sectional parameter η for sandwich section using singular vector of $\mathbf{K}^{\mathrm{A}}$.

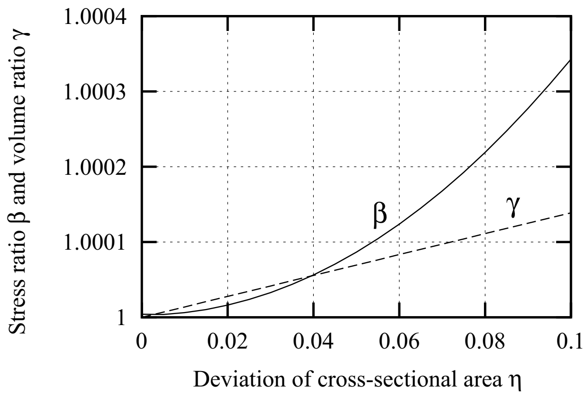


Fig. 5: Variation of the maximum stress ratio β (solid line) and the volume ratio γ (dashed line) with respect to cross-sectional parameter η for sandwich section using singular vector of $\mathbf{G}$.

Table 4: Optimization results of the frame (H-beam and box-column).

	$c = 0$	$c = 0.5$	$c = 1.0$	$c = 2.0$
A_1	11996.3	12102.1	17874.2	12101.8
A_2	16603.0	16554.9	5000.0	16555.0
A_3	5000.0	5000.0	5000.0	5000.0
A_4	17011.5	16961.2	5000.0	16961.2
V $(\times 10^9)$	4.1147	4.1147	4.0912	4.1147

this case, and the variations of β and γ with respect to η are as shown in Fig. 5. It is confirmed from Figs. 4 and 5 that approximate solutions with good accuracy can be obtained by SVD of $\mathbf{G}$.

6.3 Local search for frame with H-beam and box-column

Optimal solutions are found for the frame in Fig. 1 that consists of beams with H-section and columns with box-section. The following relations are assumed so that

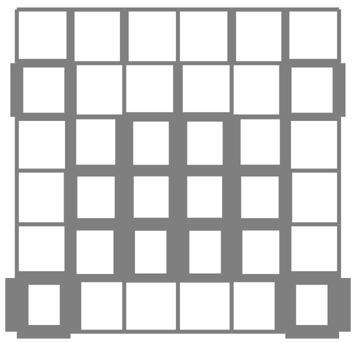


Fig. 6: Optimal solution (H-beam and box-column); $c = 1.0$.

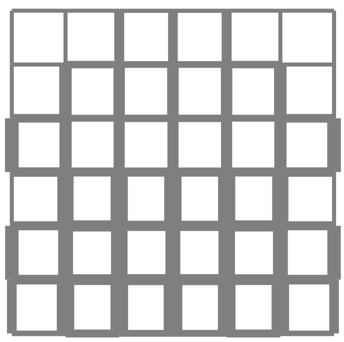


Fig. 7: Optimal solution (H-beam and box-column); $c = 0.0$.

only the cross-sectional areas are the design variables [14]:

$$\begin{cases} \text{Columns}: I_i = 1.076(A_i)^2,\ Z_i = 0.804(A_i)^{1.5} \\ \text{Beams}: \quad I_i = 3.648(A_i)^2,\ Z_i = 1.580(A_i)^{1.5} \end{cases} \tag{31}$$

The upper bound 120 is given for the displacement in x-direction of the roof level, which corresponds to 1/200 average drift angle. The lower bounds for A_i are 5000.0 for all the groups. The initial solutions are given by (27) with $A_0 = 5000$.

The optimal cross-sectional areas of the exterior columns and the optimal objective values are listed in Table 4, where the best solution has been chosen from randomly generated 10 initial solutions for each case of $c = 0.5,\ 1.0,\ 2.0$, whereas the case $c = 0.0$ is independent of the random numbers. The optimal objective value is same for $c = 0.0$, 0.5 and 2.0, but is slightly different for $c = 1.0$. The cross-sectional areas are not identical even if the objective values are same. The optimal solutions

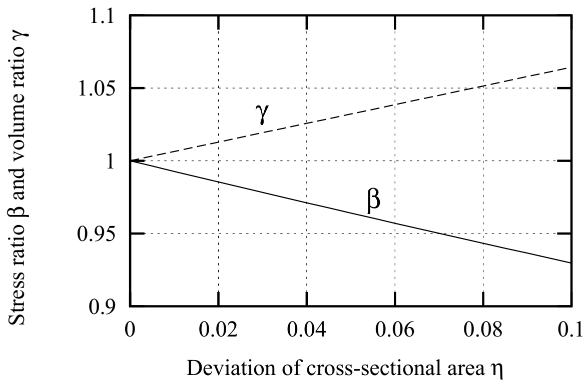


Fig. 8: Variation of the maximum stress ratio β (solid line) and the volume ratio γ (dashed line) with respect to cross-sectional parameter η for H-beam and box-column using singular vector of $\mathbf{K}^{\mathrm{A}}$ (Case 1: $h = 3$).

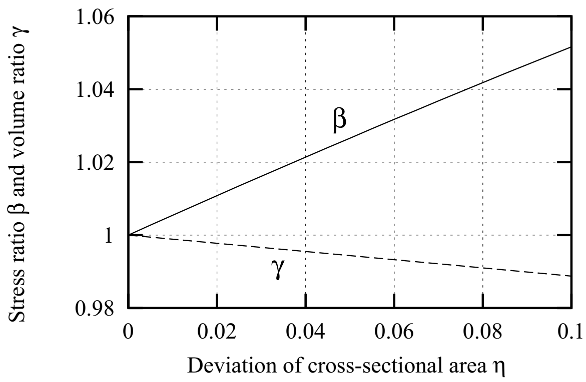


Fig. 9: Variation of the maximum stress ratio β (solid line) and the volume ratio γ (dashed line) with respect to cross-sectional parameter η for H-beam and box-column using singular vector of $\mathbf{K}^{\mathrm{A}}$ (Case 2: $h = 6$).

for $c = 1.0$ and 0.0 are shown in Figs. 6 and 7, respectively, where the distributions of the cross-sectional areas are completely different.

The optimal solutions are fully stressed, and the displacement constraint is satisfied in equality. SVD has been carried out for $\mathbf{K}^{\mathrm{A}}$ of the optimal solution in Fig. 6 for $c = 1.0$. The singular values are listed in the 3rd column of Table 3, which have larger values than those for sandwich section. Variations of β and γ with respect to the cross-sectional parameter η are as shown in Fig. 8 in the direction of $\Delta\mathbf{A}$ found from OPT2, where $h = 3$. In this case, the maximum stress ratio decreases as η is increased, because the direction that increases V has not been found. Therefore, the number of modes h is increased to 6 in solving OPT2 to obtain the relations in Fig. 9. It is observed from Figs. 4 and 9 that the frames with H-beam and box-section has less accuracy than that with sandwich section, because I_i and Z_i are nonlinear functions of A_i. The singular values for $\mathbf{G}$ are listed in the 4th column of Table 3. The approximation results using SVD of $\mathbf{G}$ are plotted in Fig. 10. Note from Figs. 9 and 10 that accuracy of the solutions is improved using $\mathbf{G}$ instead of $\mathbf{K}^{\mathrm{A}}$.

6.4 Global search for frame with sandwich section

Approximate optimal solutions are consecutively found for the upper bound of V that is 2% larger than the optimal objective value $\tilde{V}$ of the solution in Fig. 3; i.e., $\Delta V = 0.02\tilde{V}$ in (23).

OPT3 is to be solved to maximize the minimum distance from the existing approximate optimal solutions. The solutions 2–10 found consecutively are shown in Figs. 11(a)-(i). Notice that the approximate solutions with various distributions of cross-sectional areas have been successfully found. The constraints on the total structural volume is satisfied in equality in all the solutions.

The value of the minimum distance τ for solutions 2–10 are plotted in Fig. 12. It can be confirmed from Fig. 12 that τ decreases as more solutions are found. This way, approximate solutions can be globally searched.

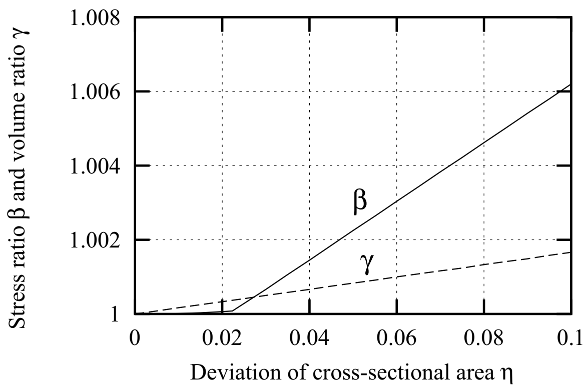
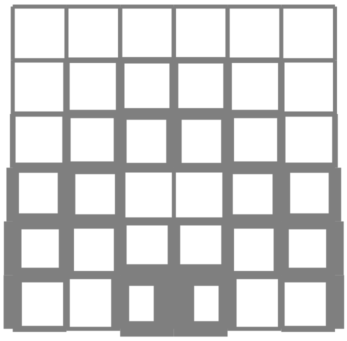
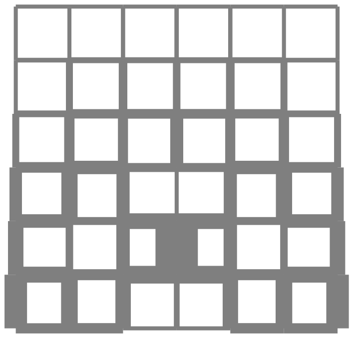


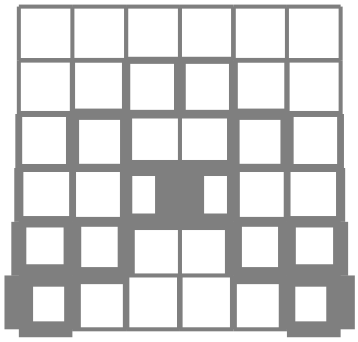
Fig. 10: Variation of the maximum stress ratio β (solid line) and the volume ratio γ (dashed line) with respect to cross-sectional parameter η for H-beam and box-column using singular vector of $\mathbf{G}$.



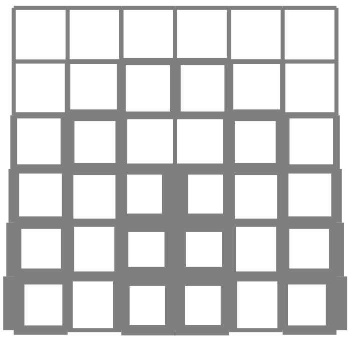
(a) Solution 2



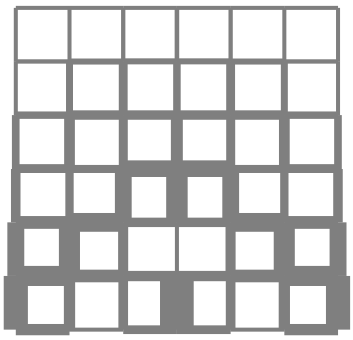
(b) Solution 3



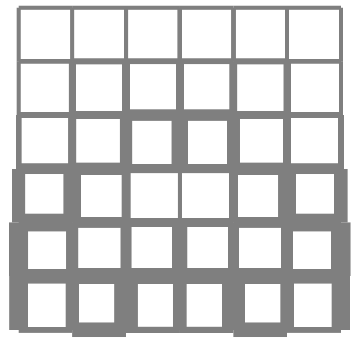
(c) Solution 4



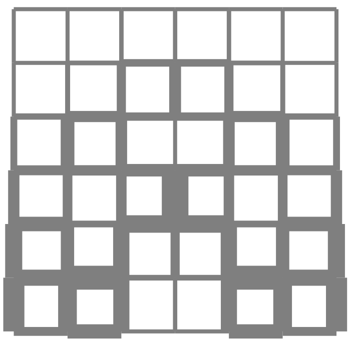
(d) Solution 5



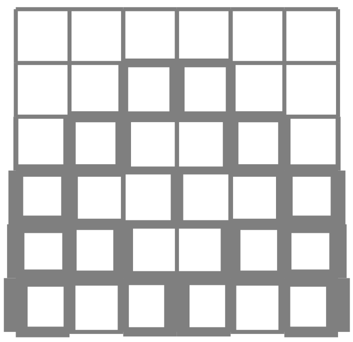
(e) Solution 6



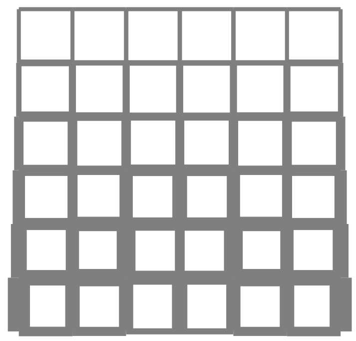
(f) Solution 7



(g) Solution 8



(h) Solution 9



(i) Solution 10

Fig. 11: Approximate optimal solutions.

It is possible to incorporate directly the preference of the designer under constraint (23) on the structural volume; e.g., the standard variation of cross-sectional areas can be minimized as shown in Fig. 13(a), or the maximum cross-sectional area can be minimized to result in Fig. 13(b), where the maximum value of A_i is 13878.8 which is about 60% of 22047.3 of the solution 1.

7 Conclusions

Methods of local and global searches of approximate optimal designs have been presented for regular frames subjected to static loads. Constraints are given for stresses and displacements.

Nonuniqueness of the optimal solution has been first demonstrated by a continuous beam with periodic boundary conditions for uniform loads. The optimal solutions can be locally searched from a solution found from an arbitrary generated initial solution. The search direction is determined by SVD of the stiffness matrix with respect to the cross-sectional areas or the sensitivity matrix of the constraints.

Approximate optimal solutions can be globally and consecutively found so as to maximize the distance from the already found solutions under stress and displacement constraints. The distance between the solutions is defined by the Euclidean norm of the differences in the cross-sectional areas. The constraint is given for the total structural volume, where the specified upper bound is slightly larger than the objective value of an optimal solution.

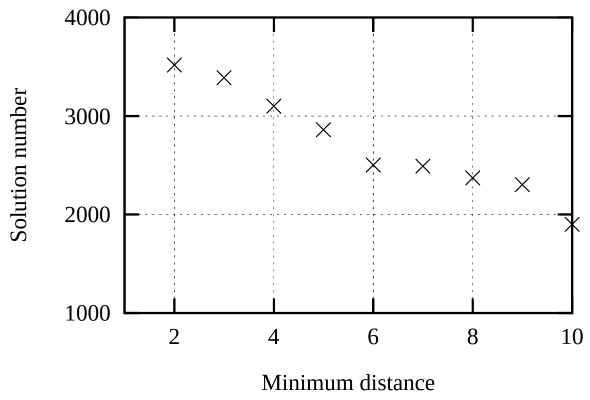


Fig. 12: Variation of minimum distance from the existing solutions.

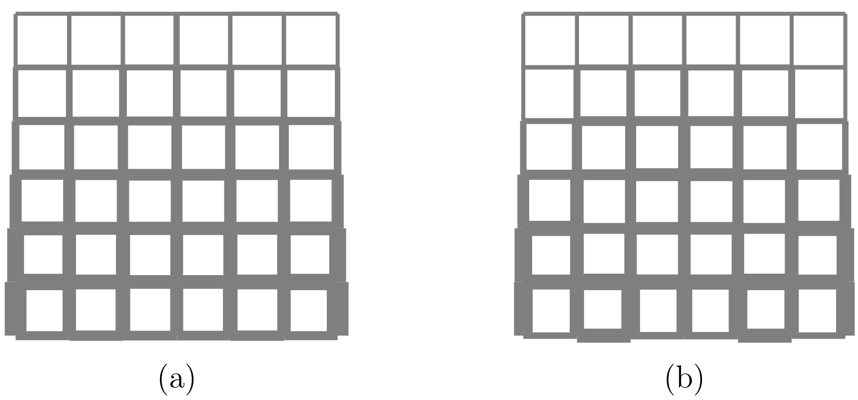


Fig. 13: Optimal solution for direct preference of cross-sectional areas; (a) minimum variation, (b) minimum maximum value.

The effectiveness of the proposed methods has been demonstrated in application to a regular plane frame. Approximate optimal solutions have been successfully found using the singular vectors of the stiffness matrix with respect to the cross-sectional areas. However, accuracy of the solutions can be improved using the singular vectors of the sensitivity matrix of the constraints.

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