Semi-Definite Programming for Topology Optimization of Trusses under Multiple Eigenvalue Constraints

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1. Abstract

The eigenvalues of free vibration as well as the linear buckling load factor are important performance measures of the structures. In this paper, the topology optimization problem for specified eigenvalue of vibration is formulated as Semi-Definite Programming (SDP), and optimal topologies are computed for several examples of plane and space trusses by applying the Semi-Definite Programming Algorithm (SDPA).

2. Keywords

Semi-Definite Programming, Topology Optimization, Multiple Eigenvalue Constraints

3. Introduction

It is well known that optimum designs for specified fundamental eigenvalue often have multiple (repeated) eigenvalues. Such an optimal structure was first presented by Olhoff and Rasmussen [1] where necessary conditions for optimality are discussed and an optimal column under buckling constraint is found by using an optimality criteria approach.

There are difficulties in optimizing distributed parameter structures for specified multiple eigenvalues, since the multiple eigenvalues are not differentiable in ordinary sense, and only directional derivatives with respect to the design variables may be calculated [2, 3].

In spite of theoretical developments for sensitivity analysis of multiple eigenvalues and optimization methods for problems under multiple eigenvalue constraints, no globally convergent algorithm seems to have been presented for optimization of large structures. Nakamura and Ohsaki [4] presented a parametric programming approach to trace a set of optimal solutions under multiple eigenvalue constraints. Although their method has been shown to be effective for bimodal case, it is very difficult to extend it to the problems with larger multiplicity of eigenvalues.

In order to overcome the difficulties due to multiplicity of eigenvalues, we present in this paper an algorithm based on the Semi-Definite Programming (SDP) which does not need explicit derivatives of eigenvalues with respect to the design variables. It is shown, in the examples, that SDPA has advantage over existing methods in view of computational efficiency and accuracy of the solutions, and an optimal topology with five-fold fundamental eigenvalue is found without any difficulty.

4. Outline of SDPA

The SDP is an extension of linear programming in a sense that in addition to linear constraints, it allows the constraints that require matrices to be positive semi-definite (notice that those constraints cannot be expressed as linear constraints). As in linear programming, interior-point methods have polynomial time worst-case complexity and perform very well in practice. SDPA [5] is a C++ implementation of a Mehrotra-type primal-dual predictor-corrector interior-point method [6, 7] for solving the standard form of SDP and its dual.

Let $R^{n \times n}$ and $\mathcal{S}^n \subset R^{n \times n}$ denote the set of all $n \times n$ real matrices and the set of all $n \times n$ real symmetric matrices, respectively. We use the notation $\mathbf{U} \bullet \mathbf{V}$ for the inner product of $\mathbf{U},\ \mathbf{V} \in R^{n \times n}$, i.e. $\mathbf{U} \bullet \mathbf{V} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{U}_{ij} \mathbf{V}_{ij}$, where $\mathbf{U}_{ij}$ and $\mathbf{V}_{ij}$ denote the (i, j)th element of $\mathbf{U}$ and $\mathbf{V}$, respectively. We write $\mathbf{X} \succeq \mathbf{O}$ and $\mathbf{X} \succ \mathbf{O}$ when $\mathbf{X} \in \mathcal{S}^n$ is positive semi-definite and positive definite, respectively. The standard form of SDP and its dual are formulated as

$$\left.\begin{array}{rl} \mathcal{P}\text{: Minimize} & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} & \sum_{i=1}^{m} \mathbf{F}_i y_i + \mathbf{X} = \mathbf{F}_0, \\ & \mathbf{X} \in \mathcal{S}^n,\ \mathbf{X} \succeq \mathbf{O}, \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll}\mathcal{D}\text{: Maximize} & \mathbf{F}_0 \bullet \mathbf{Y} \\ \text{subject to} & \mathbf{F}_i \bullet \mathbf{Y} = b_i \ (i = 1, \cdots, m), \\ & \mathbf{Y} \in \mathcal{S}^n, \ \mathbf{Y} \succeq \mathbf{O}.\end{array}\right\} \quad (2)$$

where $\mathbf{F}_i \in \mathcal{S}^n$ $(i = 0, \cdots, m)$, $\mathbf{b} \in R^m$ and $\mathbf{y} \in R^m$.

Among several softwares for SDPs that are currently available, SDPA (Semi-Definite Programming Algorithm) [5] seems to be fastest.

5. Formulation of topology optimization problem by SDP

Consider a truss with fixed locations of nodes and members that can exist. The vector of member cross-sectional areas is denoted by $\mathbf{A} = \{A_i\}$. Let $\mathbf{K}$ and $\mathbf{M}_s$ denote the stiffness matrix and the mass matrix due to the structural mass both of which are functions of $\mathbf{A}$. The mass matrix for nonstructural mass is denoted by $\mathbf{M}_0$.

The eigenvalue problem of vibration is formulated as

$$\mathbf{K}\Phi_r = \Omega_r(\mathbf{M}_s + \mathbf{M}_0)\Phi_r \ (r = 1, 2, \cdots, N^d), \quad (3)$$

where Ω_r and Φ_r are the rth eigenvalue and eigenvector, respectively, and N^d is the number of freedom of displacements. The eigenvector Φ_r is normalized by

$$\Phi_r^T(\mathbf{M}_s + \mathbf{M}_0)\Phi_r = 1 \ (r = 1, 2, \cdots, N^d). \quad (4)$$

Let $\bar{\Omega}$ denote the specified lower bound of the eigenvalues. The topology optimization problem for specified fundamental eigenvalue is formulated as

$$\left.\begin{array}{lll}\text{TOP:} & \text{Minimize} & \sum_{i=1}^{N^m} A_i L_i, \\ & \text{subject to} & \Omega_r \geq \bar{\Omega} \ (r = 1, 2, \cdots, N^d), \\ & & A_i \geq 0 \ (i = 1, 2, \cdots, N^m),\end{array}\right\} \quad (5)$$

where L_i is the length of the ith member, and N^m is the number of members. The optimal topology is obtained by removing the members with $A_i = 0$. A small positive lower bound is usually given for A_i throughout the optimization process in order to prevent instability of the structure.

Consider a structure where $\Omega_1 \geq \bar{\Omega}$ is satisfied. In this case the Rayleigh's principle leads to the following inequality for any kinematically admissible mode $\boldsymbol{\psi}$:

$$\boldsymbol{\psi}^T[\mathbf{K} - \bar{\Omega}(\mathbf{M}_s + \mathbf{M}_0)]\boldsymbol{\psi} \geq 0. \quad (6)$$

This inequality implies that the matrix $\{\mathbf{K} - \bar{\Omega}(\mathbf{M}_s + \mathbf{M}_0)\}$ is positive semi-definite, and formulations of SDP may be possible. The matrices $\mathbf{K}_i$ and $\mathbf{M}_i$ are defined as $\mathbf{K}_i = \dfrac{\partial \mathbf{K}}{\partial A_i}$, $\mathbf{M}_i = \dfrac{\partial \mathbf{M}_s}{\partial A_i}$. Since $\mathbf{K}$ and $\mathbf{M}_s$ are linear functions of A_i for trusses, those are written as

$$\mathbf{K} = \sum_{i=1}^{N^m} A_i \mathbf{K}_i, \quad \mathbf{M}_s = \sum_{i=1}^{N^m} A_i \mathbf{M}_i. \quad (7)$$

So, the primal and dual problems of SDP for this case are formulated as

$$\left.\begin{array}{lll}\mathcal{P}': & \text{Minimize} & \sum_{i=1}^{N^m} A_i L_i, \\ & \text{subject to} & \mathbf{X} = \sum_{i=1}^{N^m} (\mathbf{K}_i - \bar{\Omega}\mathbf{M}_i)A_i - \bar{\Omega}\mathbf{M}_0, \\ & & \mathbf{X} \in \mathcal{S}^{N^d}, \mathbf{X} \succeq \mathbf{O}, \\ & & A_i \geq 0 \ (i = 1, 2, \cdots, N^m).\end{array}\right\} \quad (8)$$

$$\left.\begin{array}{lll}\mathcal{D}': & \text{Maximize} & \bar{\Omega}\mathbf{M}_0 \bullet \mathbf{Y}, \\ & \text{subject to} & (\mathbf{K}_i - \bar{\Omega}\mathbf{M}_i) \bullet \mathbf{Y} \leq L_i, \\ & & \mathbf{Y} \in \mathcal{S}^{N^d}, \mathbf{Y} \succeq \mathbf{O}.\end{array}\right\} \quad (9)$$

Problems $\mathcal{P}'$ and $\mathcal{D}'$ are solved successively to find optimal solutions by using the SDPA algorithm. This method is very effective for the case of optimum designs with multiple eigenvalues. It is because sensitivity coefficients of the eigenvalues with respect to the design variables are not needed. Moreover, the SDPA incorporates data structures for handling sparse matrices and an efficient method proposed by Fujisawa *et al.* [8] for computing search directions for problems with large sparse matrices.

6. Examples

In the examples, in order to see the effectiveness of the proposed method in view of computational efficiency and accuracy of the solutions, we compare the computational results with those computed by existing parametric programming approach [9] and sequential quadratic programming algorithm [10]. The results by SDPA, PP and SQP are listed in Table 1, [11]. In the following examples, the material of the members is steel where elastic modulus E is 205.8 GPa and the mass density ρ is 7.86×10^{-3} kg/cm^2. In SDPA, E and ρ are scaled so that $E = 1000.0$ is satisfied to prevent divergence in the process of finding a feasible solution. The specified eigenvalue is 1000.0 rad^2/s^2 for all the cases. The computation has been carried out on Sun Ultra II (Ultra SPARC II 300MHz with 256 MB memory).

Plane square grids

Optimal topologies are found for plane square trusses with 2×2, 3×3, 4×4 and 5×5 grids to compare the performances of the methods. A nonstructural mass of 2.1×10^4 kg is located at the the upper-right corner for each case. A 5×5 grid is as shown in Fig. 1(a), where L_i for all the vertical and horizontal members are 200.0 cm. The optimal topology of 5×5 grid found by SDPA after removing extremely slender members with $A_i < 2.0 \times 10^{-3}$ cm^2 is as shown in Fig. 1(b), where the width of each member is proportional to its cross-sectional area. Note from Fig. 1(b) that there exists a kind of net with secondary members for preventing instability of the ten-bar truss formed by the primal members with moderately large cross-sectional areas. Those secondary members cannot be removed because the two long members, each composed of five short members, will be unstable without those members.

The multiplicity of the lowest eigenvalue is two, and the corresponding modes are as illustrated in Fig. 2. The displacements of node 9 where the nonstructural mass is located is very large in the mode (a), whereas local flexural deformation at node 1 dominates in the mode (b). The local modes such as mode (b) may be suppressed by fixing the unstable nodes 1-8. The maximum and minimum values of the cross-sectional areas of the primary members are 43.991 cm^2 and 40.566 cm^2, respectively, whereas those of the secondary members are 2.2299 cm^2 and 6.7598×10^{-3} cm^2, respectively. A practically optimal topology may be found, if necessary, by removing the secondary members and fixing the unstable nodes 1-8 in Fig. 1(b) to generate a frame with two members. Note that the node 9 is not fixed. Let γ denote the radius of gyration of each member. The fundamental eigenvalue of the frame for $\gamma = 30.0$ cm is 973.28 rad^2/s^2 which is smaller than the specified value due to the lateral deformation of two long members. The eigenvalue may be increased by assigning larger value for γ.

The results by SDPA, PP and SQP are listed in Table 1. It may be observed from these results that the performance of SDPA is better than that of PP in view of accuracy, and CPU time of SDPA is less than that of SQP. In addition to these advantages, SDPA has no difficulty in finding optimal solutions with multiple eigenvalues. Note that the difference among the second eigenvalues computed by three methods is very large, because those are sensitive to the cross-sectional areas of the secondary members. The second eigenvalues, however, are associated with local modes which are not practically important.

Since formulation of multiple eigenvalues has not been used for SQP, the optimization process has not converged if $\bar{A}_i = 0.01$ for the 5×5 grid. Therefore moderately large lower bound is needed for SQP to prevent the divergence due to the multiplicity of eigenvalues. Note again that it is not important from the practical point of view to find optimal solutions with multiple eigenvalues one of which is associated with a locally vibrating mode such as mode (b) in Fig. 2. Since CPU time for SQP will be much larger if the multiplicity of the fundamental eigenvalue is considered, the efficiency of SDPA compared with SQP has been successfully demonstrated by these examples. Positive lower bound on cross-sectional areas are also given for PP to avoid unnecessary computational cost due to multiplicity of eigenvalues corresponding to locally vibrating modes.

A double-layer grid

Consider next a double-layer grid as shown in Fig. 3(a). Nonstructural masses are located at all the upper nodes. The lengths of members in x- and y-directions are 300.0 cm and 200.0 cm, respectively, and the distance between the upper and lower planes is 200.0 cm. The truss has two planes of symmetry. The optimal topology found by SDPA after removing members with $A_i < 2.0 \times 10^{-3}$ cm^2 is as shown in Fig. 3(b).

The optimization results by SDPA is as listed in Table 1. Note that the values of five lowest eigenvalues are all equal to 1000.0 rad^2/s^2, i.e. the multiplicity of eigenvalues of the optimal solution is five, where all the eigenmodes are global modes. Symmetricity properties of the five fundamental eigenmodes are as listed in Table 2, where S and A indicate symmetric and antisymmetric, respectively. SDPA has not found any difficulty in computing an optimal solution even for such case with five-fold fundamental eigenvalue. In addition to these advantages, it is theoretically guaranteed that a symmetric solution is always found without assigning any side constraints in order to preserve symmetricity of the cross-sectional areas.

7. Conclusions

Topology optimization problem of trusses for specified eigenvalue of vibration is formulated as Semi-Definite Programming (SDP), and an algorithm is presented based on the Semi-Definite Programming Algorithm (SDPA) which fully utilizes extensively the sparseness of the matrices.

It is shown in the examples, that SDPA has advantages over existing methods in several points, e.g., computational efficiency and accuracy of the solutions. An optimal topology with five-fold fundamental eigenvalue is found without any difficulty. Note that no significant increase seems to be observed in CPU time as a result of multiplicity of eigenvalues. The proposed algorithm always finds a symmetric solution without assigning any side constraints in order to preserve symmetricity of the cross-sectional areas.

8. References

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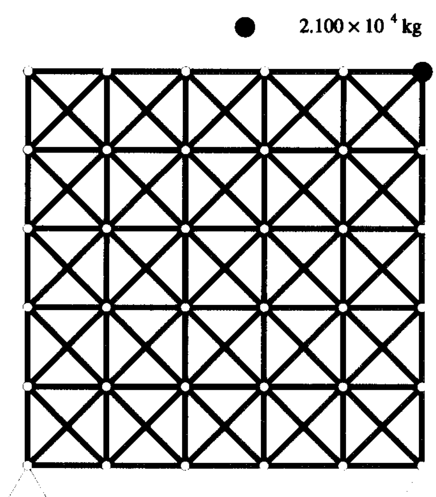
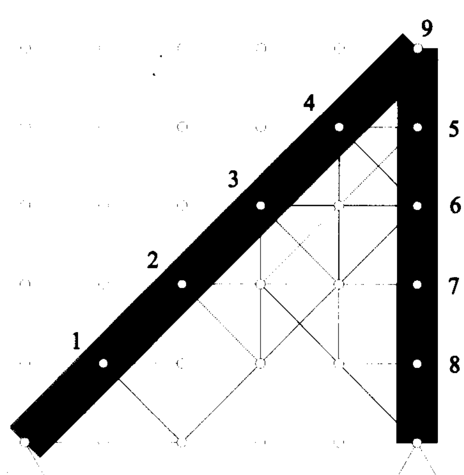
Table 1: Comparison of performances of SDPA, PP and SQP.

		SDPA [8]	PP [9]	SQP [10]
Plane square grid	Volume (cm^3)	1.6355×10^4	1.6368×10^4	1.6357×10^4
2×2	Ω_1 (rad^2/s^2)	1000.0	999.55	1000.0
($N^m = 20$)	Ω_2 (rad^2/s^2)	2145.3	5977.8	2000.3
($N^d = 14$)	$\bar{A}_i$ (cm^2)	0.0	0.01	0.001
	CPU (s)	0.11	0.83	1.21
Plane square grid	Volume (cm^3)	3.6886×10^4	3.6905×10^4	3.6890×10^4
3×3	Ω_1 (rad^2/s^2)	1000.0	999.44	1000.0
($N^m = 42$)	Ω_2 (rad^2/s^2)	1045.7	2531.6	1011.9
($N^d = 28$)	$\bar{A}_i$ (cm^2)	0.0	0.01	0.001
	CPU (s)	0.66	2.65	5.93
Plane square grid	Volume (cm^3)	6.5776×10^4	6.6126×10^4	6.5841×10^4
4×4	Ω_1 (rad^2/s^2)	1000.0	999.49	1000.0
($N^m = 72$)	Ω_2 (rad^2/s^2)	1005.0	4551.5	1903.0
($N^d = 46$)	$\bar{A}_i$ (cm^2)	0.0	0.05	0.01
	CPU (s)	2.77	5.76	15.96
	Volume (cm^3)	1.0320×10^5	1.0371×10^5	1.0446×10^5
Plane square grid	Ω_1 (rad^2/s^2)	1000.0	999.39	1000.0
5×5	Ω_2 (rad^2/s^2)	1000.0	3052.9	5219.3
($N^m = 110$)	Multiplicity	2	1	1
($N^d = 68$)	$\bar{A}_i$ (cm^2)	0.0	0.01	0.1
	CPU (s)	7.68	14.76	44.31
	Volume (cm^3)	6.4493×10^5	6.4497×10^5	
A plane arch grid	Ω_1 (rad^2/s^2)	1000.0	998.93	
($N^m = 174$)	Ω_2 (rad^2/s^2)	1000.0	999.42	
($N^d = 106$)	Multiplicity	2	2	
	$\bar{A}_i$ (cm^2)	0.0	0.01	
	CPU (s)	28.50	382.09	
	Volume (cm^3)	8.7110×10^5		
Double-layer grid	$\Omega_1, \cdots, \Omega_5$ (rad^2/s^2)	1000.0		
($N^m = 128$)	Multiplicity	5		
($N^d = 111$)	$\bar{A}_i$ (cm^2)	0.0		
	CPU (s)	25.55		



(a) A 5×5 plane square grid.



(b) Optimal topology of 5×5 grid.

Figure 1: A 5×5 plane square grid.

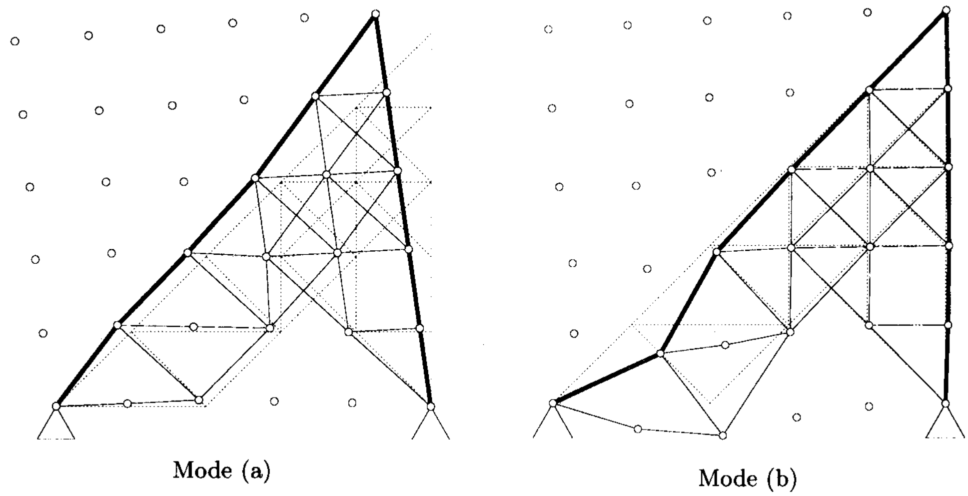
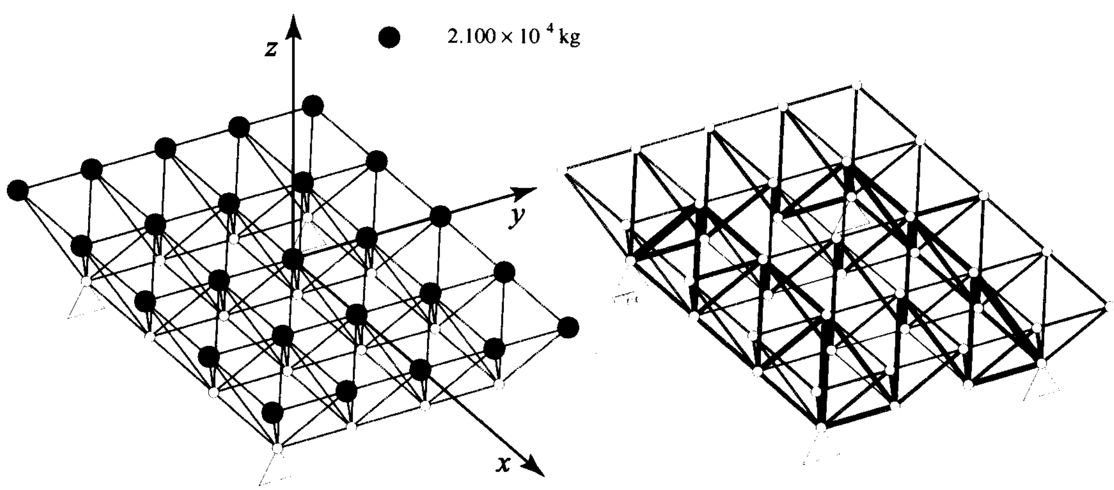


Figure 2: Eigenmodes of optimal 5 × 5 grid.



(a) A double-layer grid

(b) Optimal topology of a double-layer grid

Figure 3: A double-layer grid.

Table 2: Symmetricity of the fundamental eigenmodes of the double-layer grid.

	xz-plane	yz-plane
Mode 1	S	S
Mode 2	A	S
Mode 3	A	S
Mode 4	S	A
Mode 5	A	A