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Unified treatment of some different fabrication-cost functions in truss topology optimization

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Abstract

This paper presents a unified formulation of the fabrication costs, or the cost of nodes (connections), in truss topology optimization. This formulation is readily incorporated into an existing mixed-integer programming approach to compliance optimization with self-weight load. We perform preliminary numerical experiments to show how the optimal topology depends on different cost functions.

Keywords: Truss topology optimization, fabrication cost, mixed-integer programming, global optimization.

1 Introduction

The fabrication costs of structures can be significantly influenced by design decisions made during early stages of the design process. Therefore, it is of great importance to incorporate the fabrication cost into structural optimization. For truss topology optimization based on the ground structure approach, recent work has proposed including the fabrication cost into the problem formulation by assuming that this cost is proportional either to the number of members (bars) or to the number of nodes (connections) [1, 9, 10].

The number of existing members and the number of existing nodes are not differentiable with respect to the design variables (the member cross-sectional areas). This is the major difficulty in incorporating such a cost function into truss topology optimization, because nonsmoothness of the cost function forbids direct application of a gradient-based optimization method. As a remedy, Asadpoure *et al.* [1] proposed to make use of a regularized Heaviside function. Similarly, Torii *et al.* [10] used a negative power function. Kanno and Fujita [7] proposed an approach based on the alternating direction method of multipliers (ADMM). More heuristic manners for taking the truss design complexity into account can be found in, e.g., He and Gilbert [4] and the references therein.

In this paper, we introduce the ℓ_p-norm constraint on a vector consisting of the degrees of nodes to deal with some different fabrication-cost functions in a unified manner. Here, the degree of a node is defined as the number of members connected to the node. The presented formulation includes the cost functions proportional to the number of members [1] and the number of nodes [7, 9, 10] as two particular cases. Another interesting case is that the cost of a node is assumed to increase dramatically as the degree of a node increases. Although such a cost function seems to be realistic, to the best of the authors' knowledge it cannot be found in literature on truss topology optimization.

We incorporate the presented formulation in a straightforward manner into a *mixed-integer second-order cone programming* (MISOCP) approach to truss topology optimization considering the self-weight load [8]. We present numerical examples to illustrate how the difference in the fabrication-cost function affects optimal truss designs.

2 Degree of node and topology of truss

Following the conventional ground structure approach, consider an initial truss having m candidate members, n nodes, and d degrees of freedom of the nodal displacements. Let $x_1, \dots, x_m$ denote the member cross-sectional areas, which are the design variables to be optimized. The conventional compliance minimization problem is written as

$$\begin{aligned} &\text{minimize} && \pi(\boldsymbol{x}) && \text{(1a)} \\ &\text{subject to} && \boldsymbol{l}^\top \boldsymbol{x} \le \bar{V}, && \text{(1b)} \end{aligned}$$

where $\pi(\boldsymbol{x})$ is the compliance of the truss, l_i $(i = 1, \dots, m)$ is the undeformed member length, and $\bar{V}$ is the specified upper bound for the structural volume. To avoid presence of extremely thin and thick members, we consider the constraints

$$x_e \in \{0\} \cup [x_{\min}, x_{\max}], \quad e = 1, \dots, m, \tag{2}$$

where $x_{\min}$ and $x_{\max}$ are the specified lower and upper bounds, respectively, of member cross-sectional areas.

As for an external load, consider the sum of a fixed load and member self-weight loads. A ground structure includes, in general, some overlapping members, but we prohibit the presence of overlapping members in a final truss design. It is known that the compliance minimization problem with this setting can be recast as an MISOCP problem [8]. This MISOCP formulation uses binary design variables t_e $(e = 1, \dots, m)$ satisfying

$$\begin{aligned} t_e = 0 \quad &\Leftrightarrow \quad x_e = 0, && \text{(3a)} \\ t_e = 1 \quad &\Leftrightarrow \quad x_e \in [x_{\min}, x_{\max}]. && \text{(3b)} \end{aligned}$$

Namely, vector $\boldsymbol{t}$ represents the truss topology.

Let δ_v denote the degree of node v $(v = 1, \dots, n)$. To describe the relation between $\boldsymbol{\delta}$ and $\boldsymbol{t}$, it is convenient to use the incidence matrix of an undirected graph. Consider a corresponding relation between the structural elements and the graph elements, where a member and a node of the ground structure are regarded as an edge and a vertex of an undirected graph, respectively. The incidence matrix $B = (B_{ve}) \in \mathbb{R}^{n \times m}$ of this graph is defined by [3]

$$B_{ve} = \begin{cases} 1 & \text{if node } v \text{ is connected to member } e, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then we readily see that the relation

$$\boldsymbol{\delta} = B\boldsymbol{t} \tag{5}$$

holds.

3 Cost of nodes

Let $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ denote the cost function of a node. It is natural to assume that $\phi(0) = 0$ and $\phi(\delta_v) > 0$ ($\delta_v = 1, 2, \dots$). Moreover, for any $\delta_v = 1, 2, \dots$, we assume that ϕ satisfies one of the following four properties:

(i) $\phi(\delta_v + 1) - \phi(\delta_v) = 0$.

(ii) $\phi(\delta_v + 1) - \phi(\delta_v) = \phi(\delta_v) - \phi(\delta_v - 1)$.

(iii) $\phi(\delta_v + 1) - \phi(\delta_v) > \phi(\delta_v) - \phi(\delta_v - 1)$.

(iv) $(\phi(\delta_v + 1) - \phi(\delta_v))/(\phi(\delta_v) - \phi(\delta_v - 1)) \to \infty$.

As illustrated in Figure 1a, property (i) means that the cost is irrelevant to the degree of a node. This cost model has been often used in literature [7, 9, 10]. Figure 1b depicts property (ii). This model can be found in Asadpoure *et al.* [1]. The cost is assumed to be proportional to the degree, i.e., $\phi(\delta_v) = a\delta_v$ with a positive constant a. In several practical situations, it is more realistic to assume that the cost of a node dramatically increases as its degree increases. Property (iii) covers such situations, as illustrated in Figure 1c. For example, $\phi(\delta_v) = a\delta_v^r$ with $r > 1$ satisfies property (iii). Property (iv) corresponds to the extreme case of property (iii), and highly prohibits use of a node with a large degree: It concerns only the nodes with the largest degree in a truss design.

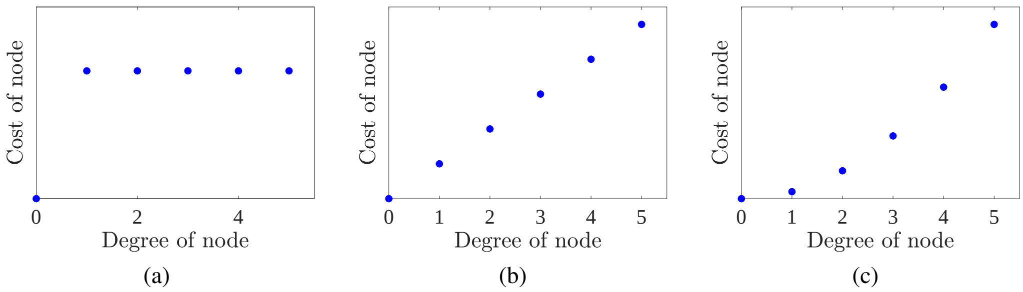


Figure 1: Cost functions of a node.

Assume that the costs of a support node and a free node are same if they have the same degree. The total cost of nodes in each case is evaluated as follows.

(i) The total cost of nodes is proportional to the number of nodes used in a truss design, i.e., the number of nonzero entries of $\boldsymbol{\delta}$, denoted $\|\boldsymbol{\delta}\|_0$. Although $\|\boldsymbol{\delta}\|_0$ is *not* a proper norm, it is often called the ℓ_0-norm of $\boldsymbol{\delta}$.

(ii) The total cost of nodes is proportional to the ℓ_1-norm of $\boldsymbol{\delta}$, i.e., $\|\boldsymbol{\delta}\|_1 = |\delta_1| + \cdots + |\delta_n|$. It is worth noting that (5) and the definition of B imply $\|\boldsymbol{\delta}\|_1 = 2\|\boldsymbol{t}\|_1$, and hence the total cost is proportional also to the number of members.

(iii) Assume that $\phi(\delta_v) = a\delta_v^2$, where $a > 0$ is a constant. Then the total cost is proportional to the square of the ℓ_2-norm of $\boldsymbol{\delta}$, i.e., $\|\boldsymbol{\delta}\|_2^2 = \delta_1^2 + \cdots + \delta_n^2$.

(iv) The total cost of nodes is a monotone function of the ℓ_∞-norm of $\boldsymbol{\delta}$, i.e., $\|\boldsymbol{\delta}\|_\infty = \max\{|\delta_1|, \dots, |\delta_n|\}$.

4 Mixed-integer programming formulation

From the observation made in section 3, we can see that the upper bound constraint on the cost of nodes is written as

$$\|\boldsymbol{\delta}\|_p \le \bar{c} \tag{6}$$

with $p = 0$, 1, 2, or ∞ and constant $\bar{c} > 0$. Our preliminary numerical experiments suggest that handling constraint (6) directly within the framework of *mixed-integer programming* (MIP) is not efficient. Specifically, when $\bar{c}$ is small, a MIP solver spends huge computational time even for small-scale problem instances. This is because, for a small value of $\bar{c}$, the optimization problem has only few feasible solutions, and hence a solver has to explore a vast number of branch-and-bound nodes before finding the first feasible solution.

As a remedy, we use the ℓ_1-exact penalty function

$$\rho \max\{\|\boldsymbol{\delta}\|_p - \bar{c}, 0\} = \min_y\{\rho y \mid y \ge \|\boldsymbol{\delta}\|_p - \bar{c},\ y \ge 0\} \tag{7}$$

with a sufficiently large penalty parameter $\rho > 0$, instead of using constraint (6). In the formulation with this penalty function, the number of feasible solutions does not depend on $\bar{c}$. Therefore, it is likely that this formulation is more suited for application of a MIP solver. In practice, we add constraint

$$\|\boldsymbol{\delta}\|_p \le y + \bar{c}, \tag{8}$$

$$y \ge 0 \tag{9}$$

to the MISOCP formulation in Kanno and Yamada [8], and add ρy to the objective function. The resulting problem is still an MISOCP problem, because we can treat constraint (8) within the framework of MISOCP as follows.

(i) $p = 0$: Analogous to Kanno and Fujita [7, section 3.2] and Kanno [6].

(ii) $p = 1$: (8) is reduced to a linear inequality constraint $\sum_{v=1}^{n} \delta_v \le y + \bar{c}$.

(iii) $p = 2$: (8) is a second-order cone constraint.

(iv) $p = \infty$: (8) is reduced to linear inequality constraints $\delta_v \le y + \bar{c}$ ($v = 1, \dots, n$).

Remark 1. It is known that, for any rational number $p \in (1, \infty)$, a constraint in the form (8) can be represented as some second-order cone constraints [2, section 3.3.1]. Therefore, any cost function in the form $\phi(\delta_v) = a\delta_v^p$ with rational $p \in (1, \infty)$ can be handled within the framework of MISOCP. ■

5 Numerical examples

The numerical examples presented in this section were computed on a 2.2 GHz Intel Core i7 processor with 8 GB RAM. We solved MISOCP problems with CPLEX ver. 12.8.0 [5]. We set parameters of CPLEX as follows. The `MIQCP strategy` parameter was two (linear programming relaxations were solved), the integrality tolerance and the relative MIP gap tolerance were 0, and the `MIP emphasis` was `BESTBOUND` (a branch-and-bound strategy emphasizing to improving the best bound value was adopted). The penalty parameter was $\rho = 10^3$. The Young's modulus and the mass density were 200 GPa and 7800 kg/m^3, respectively.

Consider the problem instance outlined in Figure 2, which shows a ground structure and fixed external load. The nodes are aligned on a 1 m × 1 m grid. This ground structure has $n = 15$ nodes and $m = 105$ members (i.e., every pair of two nodes is connected by a single member). We apply a downward vertical force of 50 kN at the bottom rightmost node, and set $x_{\max} = 2500\,\mathrm{mm}^2$ and $x_{\min} = 200\,\mathrm{mm}^2$. The upper bound for the structural volume is $\bar{V} = 1.6 \times 10^7\,\mathrm{mm}^3$.

Figure 3 shows the obtained solution for the problem without considering the cost of nodes, where the width of each member is proportional to its cross-sectional area. For the problems with the constraint on the cost of nodes, the solutions obtained by CPLEX for $p = 0$, 1, 2, and ∞ are collected in Figures 4, 5, 6,

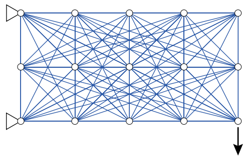


Figure 2: Problem setting of numerical examples.

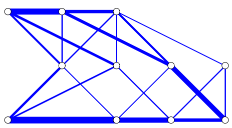
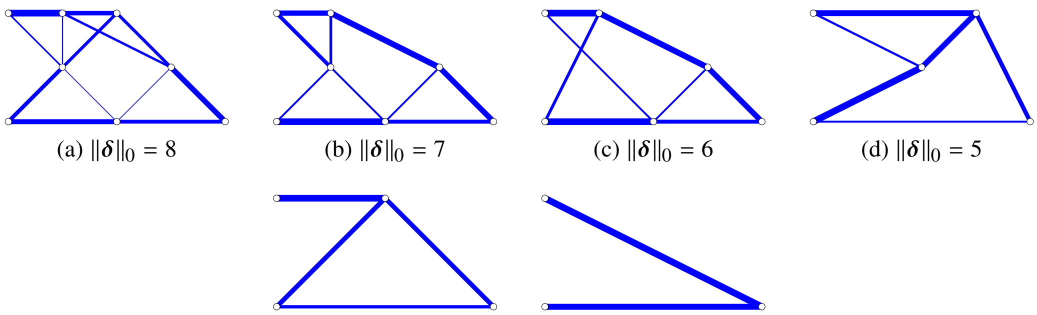


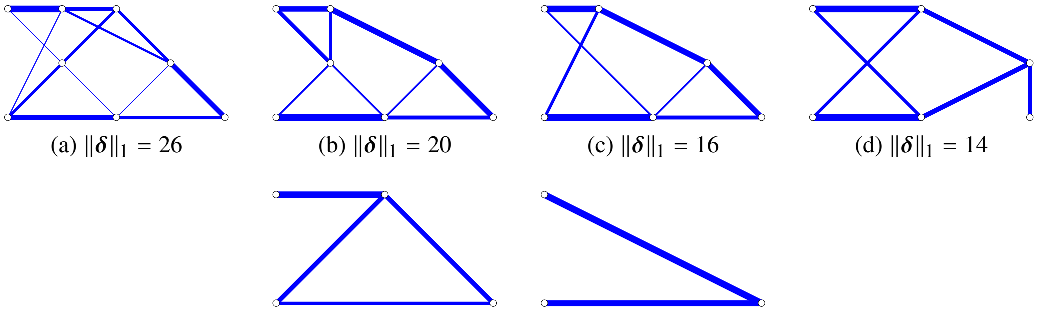
Figure 3: The obtained solution without considering the cost of nodes. $\|\boldsymbol{\delta}\|_0 = 11$, $\|\boldsymbol{\delta}\|_1 = 42$, $\|\boldsymbol{\delta}\|_2 = \sqrt{166}$ and $\|\boldsymbol{\delta}\|_\infty = 5$.



(a) $\|\boldsymbol{\delta}\|_0 = 8$ (b) $\|\boldsymbol{\delta}\|_0 = 7$ (c) $\|\boldsymbol{\delta}\|_0 = 6$ (d) $\|\boldsymbol{\delta}\|_0 = 5$

(e) $\|\boldsymbol{\delta}\|_0 = 4$ (f) $\|\boldsymbol{\delta}\|_0 = 3$

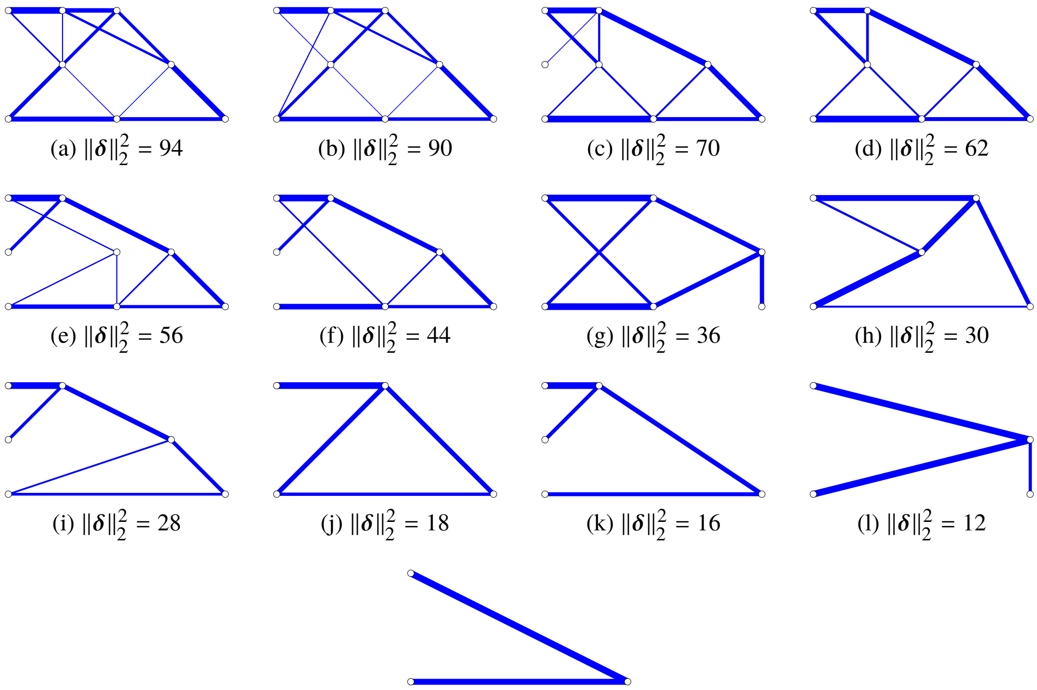
Figure 4: The obtained solutions with $p = 0$.



(a) $\|\delta\|_1 = 26$ (b) $\|\delta\|_1 = 20$ (c) $\|\delta\|_1 = 16$ (d) $\|\delta\|_1 = 14$

(e) $\|\delta\|_1 = 8$ (f) $\|\delta\|_1 = 4$

Figure 5: The obtained solutions with $p = 1$.



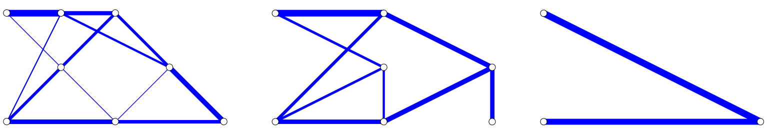
(a) $\|\delta\|_2^2 = 94$ (b) $\|\delta\|_2^2 = 90$ (c) $\|\delta\|_2^2 = 70$ (d) $\|\delta\|_2^2 = 62$

(e) $\|\delta\|_2^2 = 56$ (f) $\|\delta\|_2^2 = 44$ (g) $\|\delta\|_2^2 = 36$ (h) $\|\delta\|_2^2 = 30$

(i) $\|\delta\|_2^2 = 28$ (j) $\|\delta\|_2^2 = 18$ (k) $\|\delta\|_2^2 = 16$ (l) $\|\delta\|_2^2 = 12$

(m) $\|\delta\|_2^2 = 6$

Figure 6: The obtained solutions with $p = 2$.



(a) $\|\delta\|_\infty = 4$ (b) $\|\delta\|_\infty = 3$ (c) $\|\delta\|_\infty = 2$

Figure 7: The obtained solutions with $p = \infty$.

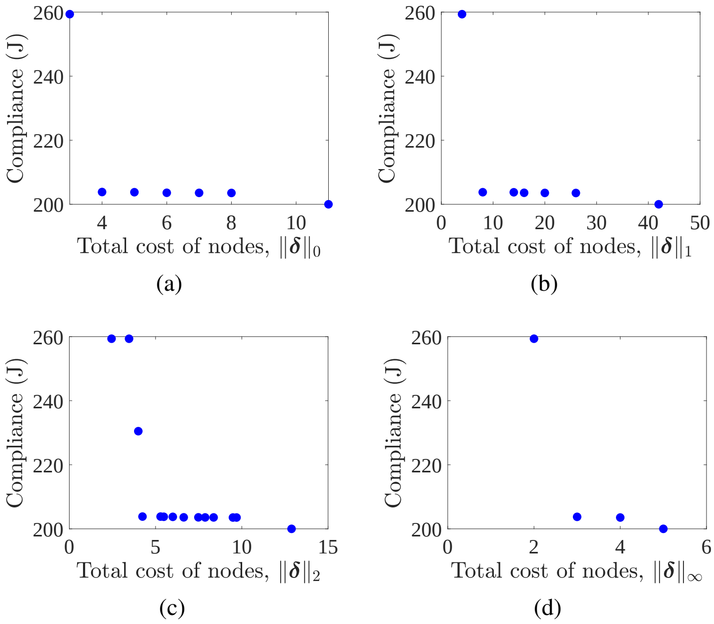


Figure 8: Variation of the compliance with respect to the cost of nodes.

and 7, respectively.[*1] We found these solutions with varying the value of $\bar{c}$. Figure 8 shows the variation of the compliance with respect to the cost of nodes. Here, the smallest compliance value corresponds to the solution shown in Figure 2, i.e., the solution without considering the cost of nodes. We can observe in Figure 8 that the cost of nodes can often be reduced at the expense of only small increase of the compliance. In contrast, the solutions shown in Figure 6k, Figure 6l, and Figure 6m have relatively large values of the compliance, as observed in Figure 8c.

Difference in the results for different cost functions is observed, for example, as follows. The solutions in Figure 4a and Figure 5a have the same number of nodes. Since the ℓ_0-norm does not take the nodal degrees into account, the Figure 4a has one node with degree five. This node is replaced with a node with degree four in Figure 5a. Thus, the sum of the degrees of nodes is reduced by adopting the ℓ_1-norm. Next, consider the solutions in Figure 5c and Figure 6f, both of which have $\|\boldsymbol{\delta}\|_1 = 16$. We see that one node with degree two in Figure 5c is divided into two nodes with degree one in Figure 6f. Thus, use of the ℓ_2-norm decreases the number of nodes with relatively large degrees, although this may possibly yield increase in the number of nodes. A similar observation can be made for the solutions in Figure 5b and Figure 6e. With the ℓ_∞-norm, the number of nodes is not taken into account. Accordingly, the solutions in Figure 7a and Figure 7b have relatively large numbers of nodes.

6 Conclusions

In this paper, we have shown that several cost functions of nodes in truss topology optimization can be expressed as a unified form of a norm constraint on the vector of the degrees of nodes. We have shown that this constraint can be incorporated into a global optimization approach based on MISOCP. In the

[*1] In Figure 5d, the bottom rightmost node is counted so that it has one degree. However, this node does not connect members. In the present formulation, no distinction between such a free node with degree one and a pin-support with degree one (e.g., the left nodes in Figure 5f) is made.

numerical examples, we have observed how the difference in the nodal cost function affects optimal truss designs.

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