# Frame model for analysis and form generation of rigid origami for deployable roof structure 

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#### Abstract

Various models and methods have been developed for design and analysis of rigid origami. Rigid origami is a transformable polyhedron which consists of rigid flat panels connected by rotational hinges. However, very few methods have been proposed for form generation of rigid origami with a simple but not regular crease pattern. It is essential to obtain a simple crease pattern in terms of constructability when we apply rigid origami to deployable structures in architecture. In addition, it is important to develop a new model that can be used consistently throughout the process of form generation and analysis of the kinematics of rigid origami. In this paper, we present the authors' recent studies and their extensions about form generation of a developable and rigid-foldable polyhedron that approximates a curved surface with a simple crease pattern using a frame model which is simple but enables us to use the same variables in form generation, evaluation of kinematic indeterminacy and large-deformation analysis using a general finite element analysis software.


Keywords: rigid origami, deployable structure, frame model, form generation, finite mechanism

## 1. Introduction

Rigid-foldable origami, or rigid origami, is a kind of polyhedral origami whose facets do not deform either in-plane or out-of-plane directions throughout the folding process. Its facets and crease lines can be substituted with rigid panels and hinges, respectively. This feature is well suited to engineering application especially for architectural deployable structures, because mechanism of rigid origami does not rely on the property of its material, i.e., it is scalable and applicable to large scale structures. Although there are some applications of the concept of origami [1] in the architectural scale, there are only few examples because of the difficulty in designing rigid-foldable mechanism.
There are some typical rigid-foldable crease patterns such as Miura-ori [2], Resch's pattern [3] and waterbomb tessellations. However, we often cannot just apply them to architectural purposes because of the various demands of the shape of the building reflecting its floor plan and its exterior design. Therefore, it is important to develop a method for form generation of a general-shaped rigid origami, or the general polyhedron which satisfies the conditions for rigid-foldability and developability or flatfoldability. Tachi $[4,5]$ formulated the geometric constraint of general quadrilateral mesh origami and presented the method for form finding of rigid-foldable quad mesh origami. Although his method is available for arbitral quad mesh origami, a known crease pattern such as Miura-ori, discrete Voss surface [6] or the hybrid surface of them is used as the initial configuration for form finding because it must satisfy the geometric condition for rigid-foldability. A typical crease pattern is also used to approximate a target surface by rigid origami; e.g., Dudte et al. [7] used generalized Miura-ori, and Zhao et al. [8] used generalized waterbomb tessellations. However, the generated crease lines tend to be the repetition of the similar pattern. It is difficult to obtain the polyhedra which have various degrees of freedom (DOFs) using a typical crease pattern.

[^0]In our resent paper [9], we proposed a method for approximating a curved surface with a rigidfoldable polyhedral origami which does not rely on a typical crease pattern. Only the developability and rigid-foldability of polyhedron are considered and the condition for flat-foldability is not incorporated. When a polyhedron is developable to a plane, the sum of angles between adjacent crease lines around each interior vertex needs to be equal to $2 \pi$. Although this necessary condition seems to be simple, it is not easy to directly obtain a shape of polyhedron satisfying this condition if a typical crease pattern is not used. Thus, we formulated an optimization problem to minimize the sum of errors of angles at the all interior vertices so that the generated polyhedron satisfies the condition for developability. Optimization starts from a triangulated curved surface to be approximated. If there isn't any unfolded or flat-folded crease line, the number of DOF of triangular mesh origami with $E_{\text {out }}$ edges on the boundary is equal to $E_{\text {out }}-3$ [5]. It is sometimes too large for the deployable structure in architecture. Therefore, to reduce the DOF, constraints are sequentially assigned so that the normal vectors of specified adjacent triangular facets are parallel. By removing (fixing) the crease lines between them, we can obtain a polyhedron that has both triangular and polygonal flat facets. Existence of infinitesimal mechanism is confirmed by singular value decomposition (SVD) of the linear compatibility matrix, and existence of finite mechanism is investigated by large-deformation analysis, because the generated polyhedron may not have a finite folding mechanism, i.e., cannot be continuously developed to a plane.

A frame model has been developed in Ref. [9] to carry out the procedure of form generation and evaluation of the mechanism of the polyhedron using the same model. The frame model enables us to use the same variables in each process, evaluate the kinematic indeterminacy by SVD in the same manner as the frame model of linkage mechanism [10], and carry out large-deformation analysis using general finite element analysis software. However, in Ref. [9], the type of surface is limited to a rectangular grid, and some optimization results are presented in Ref. [11]. In this study, we extend the results in Ref. [9] and derive the compatibility conditions of the coordinates of nodes of frame model for an arbitrarily triangulated surface. The compatibility conditions in the general form can be used for designing any specific type of polyhedron. In addition, the number of independent variables can be reduced using the compatibility conditions.


Figure 1: Frame model

## 2. Frame model

### 2.1. Models for analysis of rigid origami

Unstable truss model [12] and rotational hinge model [13] are generally used to analyze the kinematics of rigid origami. The former consists of rigid bars pin-jointed at each vertex. The positions of vertices are used as variables. It is convenient to directly use the positions of vertices of polyhedron; however, complicated configuration is often needed to restrain the deformation in out-of-plane direction if one or more polygonal facets such as quadrilateral, pentagon or hexagon are included in the model. The latter consists of rigid panels connected by hinges. The rotational angles of hinges on edges are used as variables. The angles are constrained so that a closed loop around each interior vertex cannot separate. Although folding state is easily demonstrated using the rotational hinge model, conversion of variables
is necessary for dealing with the shape of polyhedron. In addition, it is difficult to construct a finite element model for large-deformation analysis using general finite element analysis software.

In this paper, frame model [9] as shown in Fig. 1 is used for design and analysis of rigid origami. In Fig. 1, black bold lines, red solid lines and blue dotted lines represent frame element of frame model, mountain fold crease lines and valley fold crease lines, respectively. Frame elements are connected by hinges at the nodes on crease lines and rigidly connected at the nodes on facets. In case of a triangle mesh polyhedron, a node on an edge is located at its center point and a node on a facet can be arbitrarily defined, e.g., the centroid of each triangle facet in Fig. 1. If there is any polygonal flat facet, it is assumed as the combination of triangular facets and nodes are also located at the center points of the edges of these triangles. Variables for form generation and analysis of the mechanism are the coordinates of the nodes on edges of triangular facets. They satisfy the compatibility condition such that the end points of the edge shared by adjacent triangular facets meet at the same point.


Figure 2: Adjacent triangular facets

### 2.2. Compatibility conditions of frame model

In this section, we formulate the compatibility conditions of the coordinates of the nodes on edges of frame model when the polyhedron is homeomorphic to a disk. As shown in Fig. 2, let 1-5 denote the indices of the nodes and A-D denote the indices of the vertices on adjacent triangular facets. Since the node on the edge is located at its center point, the position vector $\mathbf{q}_{\mathrm{B}}$ of vertex $B$ is written as

$$
\begin{equation*}
\mathbf{q}_{\mathrm{B}}=\mathbf{r}_{5}+\mathbf{r}_{1}-\mathbf{r}_{4}=\mathbf{r}_{5}+\mathbf{r}_{2}-\mathbf{r}_{3} \tag{1}
\end{equation*}
$$

where $\mathbf{r}_{i}(i=1,2, \ldots, 5)$ represents the position vector of node $i$. The position vector of vertex $C$ is also represented by the position vectors of nodes in the same manner. Thus, the relation among the position vectors of nodes is obtained as

$$
\begin{equation*}
\mathbf{r}_{1}-\mathbf{r}_{4}=\mathbf{r}_{2}-\mathbf{r}_{3} \Leftrightarrow \mathbf{r}_{1}-\mathbf{r}_{2}+\mathbf{r}_{3}-\mathbf{r}_{4}=\mathbf{0} \tag{2}
\end{equation*}
$$

Let $x_{i}^{e}, y_{i}^{e}$ and $z_{i}^{e}$ denote $x$-, $y$ - and $z$-coordinates of the node $i$ on edge. Eq. (2) can be rewritten in the matrix form as follows:

$$
\left[\begin{array}{llll}
1 & -1 & 1 & -1
\end{array}\right]\left(\begin{array}{l}
x_{1}^{e}  \tag{3}\\
x_{2}^{e} \\
x_{3}^{e} \\
x_{4}^{e}
\end{array}\right)=0,\left[\begin{array}{llll}
1 & -1 & 1 & -1
\end{array}\right]\left(\begin{array}{l}
y_{1}^{e} \\
y_{2}^{e} \\
y_{3}^{e} \\
y_{4}^{e}
\end{array}\right)=0,\left[\begin{array}{llll}
1 & -1 & 1 & -1
\end{array}\right)\left(\begin{array}{l}
z_{1}^{e} \\
z_{2}^{e} \\
z_{3}^{e} \\
z_{4}^{e}
\end{array}\right)=0 .
$$

When the number of interior edges and all edges of the polyhedron are $E_{i n}$ and $E$, respectively, $E_{i n}$ equations similar to Eq. (3) can be obtained for all interior edges, and the compatibility conditions of the coordinates of the nodes on edges are written as

$$
\begin{equation*}
\mathbf{C}_{0} \mathbf{x}^{e}=\mathbf{0}, \mathbf{C}_{0} \mathbf{y}^{e}=\mathbf{0}, \mathbf{C}_{0} \mathbf{z}^{e}=\mathbf{0} \tag{4}
\end{equation*}
$$

where $\mathbf{x}^{e}, \mathbf{y}^{e}$ and $\mathbf{z}^{e}$ are the vectors consisting of $x_{i}^{e}, y_{i}^{e}$ and $z_{i}^{e}(i=1,2, \ldots, E)$, respectively, and $\mathbf{C}_{0}$ is the $E_{i n} \times E$ matrix whose elements are $-1,0$ or 1 . When the rank of $\mathbf{C}_{0}$ is $R$, Eq. (4) can be rewritten using the full rank $R \times E$ matrix $\mathbf{C}$ as

$$
\begin{equation*}
\mathbf{C} \mathbf{x}^{e}=\mathbf{0}, \mathbf{C} \mathbf{y}^{e}=\mathbf{0}, \mathbf{C} \mathbf{z}^{e}=\mathbf{0} \tag{5}
\end{equation*}
$$

In fact, $R$ is equal to $E-V$, and considering Euler's polyhedron theorem, $E-V=F-1$ is satisfied where $V$ and $F$ represent the number of vertices and facets of the polyhedron. The equation $R=F-1$ is inductively proved when we suppose that a triangular facet is sequentially added to the polyhedron. Therefore, the coordinates of $E-V$ nodes on edges are obtained from those of $V$ independent nodes. In addition, since the coordinates of the nodes on facets can be arbitrarily assigned, they can be represented by the linear combinations of the coordinates of the nodes on edges as

$$
\begin{equation*}
\mathbf{x}^{f}=\mathbf{D}_{f} \mathbf{x}^{e}, \mathbf{y}^{f}=\mathbf{D}_{f} \mathbf{y}^{e}, \quad \mathbf{z}^{f}=\mathbf{D}_{f} \mathbf{z}^{e} \tag{6}
\end{equation*}
$$

where $\mathbf{x}^{f}, \mathbf{y}^{f}$ and $\mathbf{z}^{f}$ are the vectors consisting of $x$-, $y$ - and $z$-coordinates of the nodes on facets, and $\mathbf{D}_{f}$ is the $F \times E$ matrix. On the other hand, the coordinates of the vertices of polyhedron can also be represented by the linear combinations of the coordinates of the nodes on edges in the same way as Eq. (6). Let $\mathbf{p}^{x}, \mathbf{p}^{y}$ and $\mathbf{p}^{z}$ denote the vectors of $x$-, $y$ - and $z$-coordinates of vertices, which are written as

$$
\begin{equation*}
\mathbf{p}^{x}=\mathbf{D}_{p} \mathbf{x}^{e}, \quad \mathbf{p}^{y}=\mathbf{D}_{p} \mathbf{y}^{e}, \quad \mathbf{p}^{z}=\mathbf{D}_{p} \mathbf{z}^{e} . \tag{7}
\end{equation*}
$$

Therefore, we can define the polyhedron and its frame model using the coordinates of the $V$ independent nodes on edges.

### 2.3. Reduction of variables

Let $\overline{\mathbf{x}}^{e}$ denotes the vector consisting of $x$-coordinates of the $V$ independent nodes on edges and $\hat{\mathbf{x}}^{e}$ denotes the vector of $x$-coordinates of the remaining $E-V$ nodes on edges. According to the first equation in Eq. (5), $\hat{\mathbf{x}}^{e}$ is written as

$$
\left[\begin{array}{ll}
\widehat{\mathbf{C}} & \overline{\mathbf{C}} \tag{8}
\end{array}\right]\binom{\hat{\mathbf{x}}^{e}}{\overline{\mathbf{x}}_{e}^{e}}=\hat{\mathbf{C}}^{\hat{\mathbf{x}}^{e}}+\overline{\mathbf{C}}^{e}=\mathbf{0} \Leftrightarrow \hat{\mathbf{x}}^{e}=-\hat{\mathbf{C}}^{-1} \overline{\mathbf{C}}^{-e}
$$

where $\overline{\mathbf{C}}$ and $\widehat{\mathbf{C}}$ are the matrices whose columns are the columns of $\mathbf{C}$ corresponding to $\overline{\mathbf{x}}^{e}$ and $\hat{\mathbf{x}}^{e}$, respectively, and $\widehat{\mathbf{C}}$ is a $(E-V) \times(E-V)$ regular matrix. Similar equation as Eq. (8) can be formulated for $y$ - and $z$-coordinates of the nodes on edges. From Eqs. (6), (7) and (8), the coordinates of all the nodes of frame model and the vertices of polyhedron can be obtained from the coordinates of independent nodes on edges as follows:

$$
\begin{gather*}
\mathbf{x}=\left(\begin{array}{c}
\overline{\mathbf{x}}^{e} \\
\hat{\mathbf{x}}^{e} \\
\mathbf{x}^{f}
\end{array}\right)=\left[\begin{array}{c}
\mathbf{I}_{V} \\
-\hat{\mathbf{C}}^{-1} \overline{\mathbf{C}} \\
\mathbf{D}_{f}\left[\begin{array}{c}
\mathbf{I}_{V} \\
-\hat{\mathbf{C}}^{-1} \overline{\mathbf{C}}
\end{array}\right]
\end{array}\right] \overline{\mathbf{x}}^{e}=\mathbf{G} \overline{\mathbf{x}}^{e}, \mathbf{y}=\left(\begin{array}{c}
\overline{\mathbf{y}}^{e} \\
\hat{\mathbf{y}}^{e} \\
\mathbf{y}^{f}
\end{array}\right)=\mathbf{G} \overline{\mathbf{y}}^{e}, \mathbf{z}=\left(\begin{array}{c}
\overline{\mathbf{z}}^{e} \\
\hat{\mathbf{z}}^{e} \\
\mathbf{z}^{f}
\end{array}\right)=\mathbf{G} \overline{\mathbf{z}}^{e}  \tag{9}\\
\mathbf{p}^{x}=\overline{\mathbf{D}}_{p} \overline{\mathbf{x}}^{e}, \mathbf{p}^{y}=\overline{\mathbf{D}}_{p} \overline{\mathbf{y}}^{e}, \mathbf{p}^{z}=\overline{\mathbf{D}}_{p} \overline{\mathbf{z}}^{e} \tag{10}
\end{gather*}
$$

where $\overline{\mathbf{D}}_{p}$ is the matrix whose columns are the columns of $\mathbf{D}_{p}$ corresponding to the independent components of the coordinates. Therefore, we can reduce the number of variables using Eqs. (9) and (10).
$\overline{\mathbf{C}}$ and $\widehat{\mathbf{C}}$ can be obtained from $\mathbf{C}_{0}$ by using the algorithm presented by Zhang et al. [14]. The reduced row-echelon form (RREF) [15] is used to effectively specify the independent set of nodes. The RREF of any finite matrix can be defined by the sequence of elementary row operations, and it has the following properties:
(a) The first nonzero element in any nonzero row is 1.
(b) In any column which contains the leading 1 of each nonzero row, 1 appears only once and all the other elements are 0 .
(c) Because of (a) and (b), the columns which contains such leading 1 are independent of each other.
(d) The RREF saves the rank of any matrix. It is equal to the number of columns which contains such leading 1.
Thus, the RREF of $\mathbf{C}_{0}$ contains $E-V$ independent columns which has the leading 1 of the nonzero rows. These columns correspond to $\widehat{\mathbf{C}}$ and the remaining columns correspond to $\overline{\mathbf{C}}$.

## 3. Form generation

The procedure of form generation of a developable and rigid-foldable polyhedron is shown in Fig. 3. The target surface is defined using a Bézier surface. Some polyhedra which have different DOFs can be generated by optimization, and we choose the best solution considering the result of the analysis of the finite mechanism. If there is no suitable solution, a different pattern of triangulation is tried [9].


Figure 3: Flowchart for generating a rigid-foldable polyhedron [9]
Let $\mathbf{X}=\left(\overline{\mathbf{x}}^{e \top}, \overline{\mathbf{y}}^{e \top}, \overline{\mathbf{z}}^{e \top}\right)^{\top}$ denote the vector of the coordinates of independent nodes on edges, which are the design variables of the optimization problem. The condition for developability of the polyhedron with set of interior vertices $V_{i n}$ is written as

$$
\begin{equation*}
F_{1}(\mathbf{X})=\sum_{v \in V_{\nu i n}}\left(\sum_{k=1}^{f_{v}} \theta_{v, k}(\mathbf{X})-2 \pi\right)^{2}=0 \tag{11}
\end{equation*}
$$

where $\theta_{v, k}(\mathbf{X})$ is the $k$ th angle between crease lines around the $v$ th interior vertex and $f_{v}$ is the number of crease lines connected to the $v$ th interior vertex. To reduce the DOF, a polyhedron with polygonal flat facets is generated by assigning a condition to make specified pairs of normal vectors of adjacent triangle facets to be parallel, and by removing the crease lines between them. Let $E_{D}$ represent a set of crease lines to be removed. The condition for removing crease lines is written as

$$
\begin{equation*}
F_{2}(\mathbf{X})=\sum_{e \in E_{D}}\left\|\mathbf{n}_{e, 1}(\mathbf{X}) \times \mathbf{n}_{e, 2}(\mathbf{X})\right\|^{2}=0 \tag{12}
\end{equation*}
$$

where $\mathbf{n}_{e, k}(k=1,2)$ are unit normal vectors of facets connected to the $e$ th crease line. Since $F_{1} \geq 0$ and $F_{2} \geq 0$ are satisfied for any $\mathbf{X}$, we minimize the sum of them; $F(\mathbf{X})=F_{1}(\mathbf{X})+F_{2}(\mathbf{X})$ and when it converges to approximately zero, optimization is regarded as successful.

The difference between $z_{i}^{e}(\mathbf{X})$ and the $z$-coordinate of the projected point of node $i$ onto the target surface along with $z$-axis is denoted by $\Delta z_{i}^{e}(\mathbf{X})$. The upper bound of the absolute value of $\Delta z_{i}^{e}(\mathbf{X})$ is represented by $\Delta \bar{z}_{i}^{e}$. Throughout the process of optimization, the outline of the polyhedron projected to $x y$-plane is constrained not to deform and the vertices at the corners of the polyhedron are constrained not to move. Therefore, the $x$-coordinates $p_{v}^{x}(\mathbf{X})$ of vertices $v\left(v \in V_{y}\right)$ on the $y$ directional boundary of the polyhedron, the $y$-coordinates $p_{v}^{y}(\mathbf{X})$ of vertices $v\left(v \in V_{x}\right)$ on the $x$ directional boundary of the polyhedron and the $z$-coordinates $p_{v-x}^{z}(\mathbf{X})\left(v \in V_{c}\right)$ of the vertices at the corners of the polyhedron are constrained to the specified values $\bar{p}^{x}, \bar{p}^{y}$ and $\bar{p}^{z}$ respectively, where $V_{y}, V_{x}$ and $V_{c}$ are the sets of vertices on the $y$-directional boundaries, the $x$-directional boundaries and at the corners of the polyhedron, respectively. The upper bounds and lower bounds of $\theta_{v, k}(\mathbf{X})$ are denoted by $\theta_{\max }$ and $\theta_{\min }$. The optimization problem is written as a nonlinear programming (NLP) problem as follows:

$$
\begin{array}{lll}
\text { Minimize: } & F(\mathbf{X})=F_{1}(\mathbf{X})+F_{2}(\mathbf{X}) & \\
\text { subject to: } & \theta_{\min } \leq \theta_{v, k}(\mathbf{X}) \leq \theta_{\max } & \left(v \in V_{i n} ; k=1,2, \ldots f_{v}\right) \\
& -\Delta z_{i}^{e} \leq z_{i}^{e}(\mathbf{X}) \leq \Delta \bar{z}_{i}^{e} & (i=1,2, \ldots, E)  \tag{13}\\
& p_{v}^{x}(\mathbf{X})=\bar{p}_{v}^{x} & \left(v \in V_{y}\right) \\
& p_{v}^{y}(\mathbf{X})=\bar{p}_{v}^{v} & \left(v \in V_{x}\right) \\
& p_{v}^{z}(\mathbf{X})=\bar{p}_{v}^{z} & \left(v \in V_{c}\right)
\end{array}
$$

In the same manner as in Ref. [9], the crease lines to be removed are sequentially chosen. First, the optimization problem is solved with $E_{D}$ empty. Let $\gamma_{e}$ denote the dihedral angle between the adjacent facets sharing $e$ th edge representing the crease line. The edge corresponding to the smallest difference of $\gamma_{e}$ to $\pi$ is added to $E_{D}$. Then, we solve the optimization problem (13) again.

## 4. Examples

Optimization problem (13) is solved using SLSQP (Sequential Least SQuares Programming) for NLP available in the Python library SciPy. Kinematic indeterminacy is calculated using SVD in NumPy of a Python program, and large-deformation analysis is carried out using Abaqus 2016.
Two examples are shown in Fig. 5 (Model 1) and Fig. 6 (Model 2). The initial shapes of them are shown in Fig. 4. The parameters for optimization are set as $\theta_{\max }=2 \pi / 3, \theta_{\min }=\pi / 6$ and $\Delta \overline{z_{i}}=0.25$. The number of independent nodes of frame model is 27 , which is equal to the number of vertices of the polyhedron. The results of optimization as well as the kinematic indeterminacy (DOF) and the error in large-deformation analysis are summarized in Table 1. The generated polyhedra can be regarded to satisfy the conditions of developability and existence of polygonal facets with good accuracy. It has been confirmed by the large-deformation analysis that both examples can be continuously developed to a plane without deformation of their facets, i.e. they are rigid-foldable.


Figure 4: Initial shapes


Figure 5: Model 1


Figure 6: Model 2
Table 1: Result of optimization, evaluation of DOF and large-deformation analysis

|  | Model 1 | Model 2 |
| :--- | ---: | ---: |
| $F(\mathbf{X})$ | $1.918 \times 10^{-11}$ | $1.858 \times 10^{-11}$ |
| Max. error of $\sum \theta_{v, k}$ from 360 (deg.) | $1.319 \times 10^{-4}$ | $1.307 \times 10^{-4}$ |
| Max. error of $\gamma_{e}\left(e \in E_{D}\right)$ from 180 (deg.) | $1.251 \times 10^{-4}$ | $9.573 \times 10^{-5}$ |
| Max. $\left\|\bar{z}_{i}^{e}\right\|$ | 0.250 | 0.250 |
| DOF | 5 | 5 |
| Max. absolute value of strain throughout <br> the large-deformation analysis | $1.875 \times 10^{-5}$ | $3.509 \times 10^{-5}$ |

## 5. Conclusion

The conclusion of this paper is summarized as follows:
a) The compatibility condition of the coordinates of the nodes of frame model has been formulated so that the end points of the polyhedron's edge shared by adjacent triangular facets meet at the same point. The formulation is independent of the type and method of triangulation, and can be systematically incorporated as the constraints of optimization problem for form generation of rigid-foldable origami surface.
b) The number of variables can be reduced using the compatibility condition of frame model. The independent set of nodes is effectively specified using RREF of the compatibility matrix. It has been shown that the number of independent nodes is equal to the number of vertices of the polyhedron.
c) It has been confirmed by the numerical examples that the matrix $\overline{\mathbf{C}}$ and $\widehat{\mathbf{C}}$ can be obtained effectively using the method proposed by Zhang et al. [14] and the number of independent nodes of frame model is as same as the polyhedron's vertices. We carried out optimization successively to reduce the DOF of the mechanism, generated a polyhedral surface with a heptagonal facet.

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