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OPTIMIZATION APPROACH TO FORM GENERATION OF RIGID-FOLDABLE ORIGAMI FOR DEPLOYABLE ROOF STRUCTURE

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Abstract
A method is presented for generating a deployable and rigid-foldable polyhedron with a simple and symmetric crease pattern that approximates a target surface. We formulate an optimization problem to obtain a polyhedron that satisfies the conditions for developability and symmetry. To reduce the degree of freedom of the mechanism by removing several crease lines, constraints are sequentially assigned. It is shown that a polyhedron with smaller degree of freedom can be found by assigning symmetry condition and fixing the symmetrically located crease lines simultaneously. By contrast, the optimal shape diverts from symmetric shape, if a single crease line is fixed after each process of optimization.
Keywords: *Rigid-foldable origami, Deployable structure, Form generation, Large-deformation analysis*

1. Introduction

Rigid-foldable origami (rigid origami) has the significant characteristics suitable for engineering application such as deployable structure or retractable building envelop [1]. Rigid origami is one branch of the polyhedral origami which consists of rigid flat panels connected by hinges. Its panels do not deform both in-plane and out-of-plane directions throughout the folding process. This feature is important especially in designing architectural deployable structures, because mechanism of rigid origami does not depend on its material; it is scalable and applicable to large scale structures. In practice, there are various restrictions on the shape of building because of its floor plan and its exterior design. Therefore, the method for form generation of a general rigid-foldable polyhedron is important. There are various approaches to design and analysis of rigid origami. A typical crease pattern is often used particularly in a context of approximating a target shape by rigid origami; e.g., Dudte *et al*. [2] used generalized Miura-ori, and Zhao *et al*. [3] used generalized waterbomb tessellations. However, it is difficult to obtain general crease patterns and the polyhedra which have various degree of freedom (DOF) using typical crease patterns. On the other hand, Tachi [4] developed an algorithm and a software for finding a folding pattern of a given polyhedron. Later, Demaine and Tachi [5] modified the algorithm and their method to obtain a folding pattern of any polyhedron. However, the crease pattern generated by their method is often too complicated to apply to an architectural deployable structure. Thus, it is desirable to develop a method to generate a simple but not typical crease pattern.

In our resent paper [6], we proposed a method for generating a rigid-foldable polyhedron with a general and simple crease pattern that approximates a curved surface. The proposed method yields a polyhedron which has triangle and quadrilateral facets. The developability and rigid-foldability of polyhedron are considered; however, the condition for flat foldability is not incorporated. Although the geometric necessary condition for developability is well known, it is difficult to directly obtain a shape of polyhedron satisfying the condition. Thus, we formulated an optimization problem to minimize the geometric errors using a frame model. Since the generated polyhedron that satisfies the necessary condition possibly cannot be continuously developed to a plane without deformation of its facets, kinematic indeterminacy of the mechanism is evaluated and large-deformation analysis is carried out to check developability. A frame model we proposed in the paper [6] enables us to use the same variables in form generation, evaluation of kinematic indeterminacy, and large-deformation analysis using a general finite element analysis software.

In this paper, we review our method and consider the symmetry of the polyhedron that is not considered in our previous study [6]. The optimization problem is re-formulated to satisfy symmetry conditions with respect to two planes. It is shown in the numerical examples that the number of variables is reduced and the polyhedral shapes suitable for architectural roofs can be obtained by assigning symmetry conditions.

2. Form generation of rigid-foldable polyhedron

2.1 Procedure

The procedure of form generation of a rigid-foldable polyhedron is shown in Fig. 1. Form generation starts from triangulated curved surface. The target surface is defined using a Bézier surface. Although the method for triangulation is independent of the formulation of optimization problem for form generation, we show a method to triangulate a curved surface in a form of grid as an example. We can obtain some polyhedra which have different DOFs in the optimization step and we select the best solution considering the result of large deformation analysis for evaluation of developability of generated polyhedra. If there is no suitable solution, a different pattern of triangulation is tried.

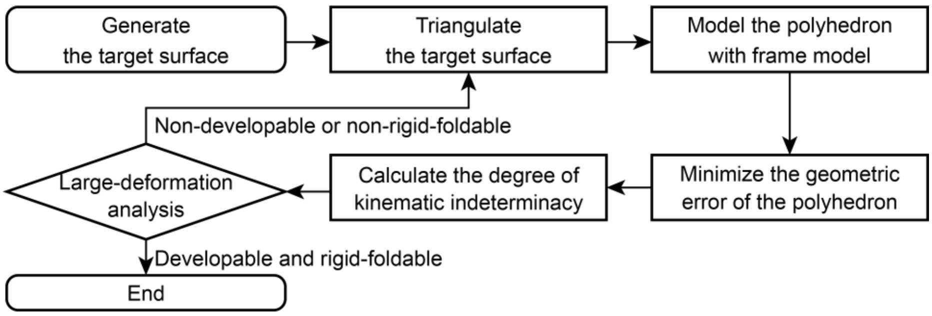


Figure 1. Flowchart for generating a rigid-foldable polyhedron [6]

2.2 Frame model

Kinematics of rigid origami is often modeled by the unstable truss model or the rotational hinge model. The former represents the shape of polyhedral origami by the coordinates of vertices. This model is used by Schenk and Guest [7] for kinematic analysis and stiffness analysis of origami. The drawback of this model is that its configuration tends to be complex to restrain the out-of-plane deformation of facets when the model has polygonal facets such as quadrilateral, pentagon or hexagon. The latter represents the shape of origami by the rotational angles of hinges on edges. Constraints of angles are assigned so that a closed loop of facets around each inner vertex cannot separate. This model is used by Tachi [8] to simulate the folding process of rigid origami. Folding state is easily investigated using this model. However, it is difficult to develop a finite element model for large-deformation analysis using a general software package.

A frame model is developed for design and analysis of rigid origami using the same variables throughout the procedure of form generation and evaluation of a mechanism. Frame model represents the shape of polyhedral origami by the coordinates of nodes on edges as variables throughout the procedure. As shown in Fig. 2, each frame element connects the node on a crease line or an outer edge and the node in a facet. Frame elements are connected by hinges on crease lines and rigidly connected in facets. A node on an edge is located at its middle point and a node on a facet can be arbitrarily defined. Coordinates of nodes on edges satisfy the constraints such that the end points of the edge shared by adjacent triangle facets meet at the same point. Thus, the number of variables are reduced. Using the frame model, we can easily model a polyhedron which contains polygons, evaluate kinematic indeterminacy in the same manner for partially rigid frames [9], and carry out large-deformation analysis using a general finite element analysis software.

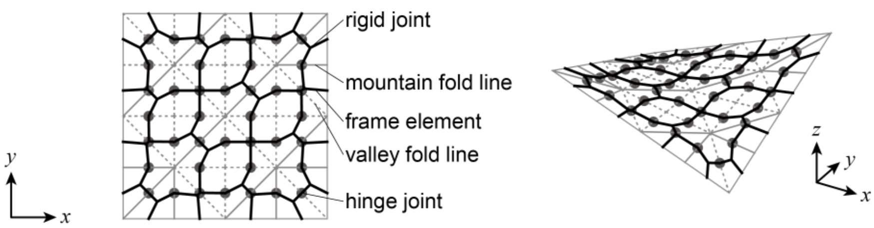


Figure 2. Frame model

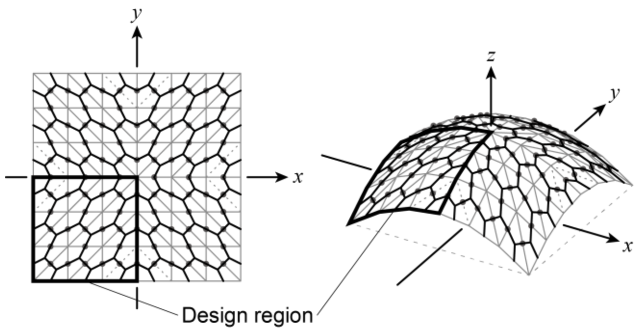


Figure 3. Initial shape and design region

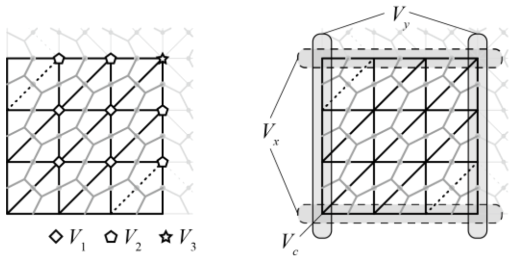


Figure 4. Classification of vertices in design region

2.3 Optimization problem for symmetric model

An optimization method is used for obtaining the configuration of the polyhedron which satisfies the condition for developability. When a polyhedron is developable to a plane, the sum of angles between adjacent crease lines around each interior vertex is equal to 2π. Optimization starts form triangulation of the target curved surface in a form of grid. As shown in Fig. 3, triangulated surface has the vertical, horizontal and diagonal edges. When the polyhedron satisfies symmetry conditions with respect to the *xz*-plane and *yz*-plane, the surface can be designed only considering one of four equal parts, e.g., the region satisfying $x \le 0$ and $y \le 0$ as shown in Fig. 3. Therefore, we formulate the optimization problem so that the quarter of the whole polyhedron Ψ satisfies the condition for developability. Let Ω denote the design region, $\mathbf{X}$ denote the coordinates of independent nodes in Ω, $\theta_{v,k}(\mathbf{X})$ denote the *k*th angle between crease lines around the vertex *v* in Ω, and f_v represent the number of crease lines connected to vertex *v* in Ω. The condition for developability of the polyhedron is written as

$$F_1(\mathbf{X}) = \sum_{v \in V_1} \left(\sum_{k}^{f_v} \theta_{v,k}(\mathbf{X}) - 2\pi \right)^2 + \sum_{v \in V_2} \left(\sum_{k}^{f_v} \theta_{v,k}(\mathbf{X}) - \pi \right)^2 + \sum_{v \in V_3} \left(\sum_{k}^{f_v} \theta_{v,k}(\mathbf{X}) - \frac{\pi}{2} \right)^2 = 0 \tag{1}$$

where V_1 is a set of inner vertices in Ω, V_2 is a set of vertices on the boundary of Ω but not at the corner of Ω or on the outer edges of Ψ, and V_3 consists of the vertex at the center of Ψ, as shown in Fig. 4.

DOF of the rigid-foldable polyhedron which has only triangle facets and no holes is $E_\mathrm{o} - 3$, where E_o is the number of outer edges [10]. This is sometimes too large for deployable structures. Therefore, constraints are assigned to reduce the DOF. A polyhedron with quadrilateral flat facets is generated by assigning condition so that the specified pair of adjacent triangle facets have parallel normal vectors, and by removing the crease lines between them. We choose the crease lines to be removed from the diagonal ones. Let E_D represent a set of crease lines to be removed. When $\mathbf{n}_{e,k}(\mathbf{X})$ $(k = 1, 2)$ are unit normal vectors of facets connected to the crease line *e* in Ω, the condition for removing specified crease lines is written as

$$F_2(\mathbf{X}) = \sum_{e \in E_\mathrm{D}} \left\| \mathbf{n}_{e,1} \times \mathbf{n}_{e,2} \right\|^2 = 0 \tag{2}$$

Since $F_1 \ge 0$ and $F_2 \ge 0$ are satisfied, the condition for developability and reducing DOF is formulated as

$$F(\mathbf{X}) = F_1(\mathbf{X}) + F_2(\mathbf{X}) = 0 \tag{3}$$

We minimize $F(\mathbf{X})$ and when $F(\mathbf{X})$ converge to approximately zero, optimization is regarded as successful.

Let $z_i(\mathbf{X})$ denote the *z*-coordinate of node *i* on the edge of the polyhedron. The difference between $z_i(\mathbf{X})$ and the *z*-coordinate of the projected point of node *i* onto the target surface along with *z* axis is defined as $\Delta z_i(\mathbf{X})$. The upper bound of the absolute value of $\Delta z_i(\mathbf{X})$ is denoted by $\Delta \overline{z}_i(\mathbf{X})$. Throughout the process of optimization, the outline of Ω projected to *xy*-plane is constrained not to deform so that the symmetry of Ψ and its projected shape on *xy*-plane are maintained. Therefore, the *x*-coordinates $p_v^x(\mathbf{X})$ of vertices *v* $(v \in V_y)$ on the *y*-directional boundary of Ω and the *y*-coordinates $p_v^y(\mathbf{X})$ of vertices *v* $(v \in V_x)$ on the *x*-directional boundary of Ω are constrained to the specified values $\overline{p}_v^x$ and $\overline{p}_v^y$, respectively, where V_y and V_x are the sets of vertices on the *y*-directional and *x*-directional boundaries of Ω, respectively. In addition, the *z*-coordinates $p_v^z(\mathbf{X})$ $(v \in V_c)$ of the vertex at the corners of Ψ are constrained to the specified value $\overline{p}_v^z$, where V_c consists of the vertex at the corner of Ψ. The upper bounds and lower bounds of $\theta_{v,k}(\mathbf{X})$ are denoted by $\theta_{\max}$ and $\theta_{\min}$. The optimization problem for minimizing the geometric error of polyhedron is written as a nonlinear programming (NLP) problem as follows:

$$
\begin{aligned}
\text{Minimize:} \quad & F(\mathbf{X}) = F_1(\mathbf{X}) + F_2(\mathbf{X}) \\
\text{subject to:} \quad & \theta_{\min} \le \theta_{v,k}(\mathbf{X}) \le \theta_{\max} \quad (v \in \Omega\, ; k = 1, \cdots f_v) \\
& -\Delta \bar{z}_i \le \Delta z_i(\mathbf{X}) \le \Delta \bar{z}_i \quad (i \in \Omega) \\
& p_v^x(\mathbf{X}) = \bar{p}_v^x \quad (v \in V_y) \\
& p_v^y(\mathbf{X}) = \bar{p}_v^y \quad (v \in V_x) \\
& p_v^z(\mathbf{X}) = \bar{p}_v^z \quad (v \in V_c)
\end{aligned}
\tag{4}
$$

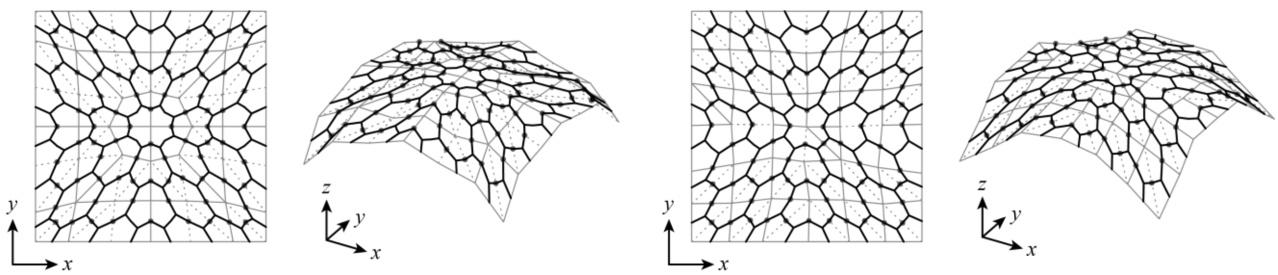
The crease lines to be removed are sequentially chosen. In the first step, we solve the optimization problem with E_{D} empty. In following step, crease lines are sequentially removed. Let γ_e denote the dihedral angle between the adjacent facets sharing *e*th edge representing the crease line. The edge corresponding to the smallest difference of γ_e to π is added to E_{D}, i.e. four crease lines of Ψ are simultaneously removed. Then, we solve the optimization problem (4) again. This process is continued until there exists no crease line to be removed or DOF is reduced to 1. Finally, we select the suitable optimal solution considering the result of evaluation of kinematic indeterminacy and large-deformation analysis.

3. Example

Optimization problem (4) is solved using SLSQP (Sequential Least SQuares Programming) for NLP available in the Python library Scipy. Gradients of the objective function and constraint functions are estimated by finite difference approximation. Kinematic indeterminacy is calculated using singular value decomposition in NumPy of a Python program, and large-deformation analysis is carried out using ABAQUS 2016.

The target surface has positive Gaussian curvature and the triangulated initial shape is shown in Fig. 3. The span of Ψ is 3 and the height of Ψ is 1.2, where the units are omitted for brevity. The parameters of optimization are set as $\theta_{\max} = 2\pi/3$, $\theta_{\min} = \pi/6$, $\Delta \bar{z}_i = 0.2$. We have obtained 6 symmetric optimal shapes whose DOFs range from 1 to 21. The 9-DOF polyhedron which includes 12 quadrilateral facets as shown in Fig. 5 is selected as an example to be investigated. The asymmetric model which has the same DOF is also generated for comparison. In the case of asymmetric model, one crease line is removed in each optimization step. In addition, the 5-DOF symmetric polyhedron is also investigated, while a 5-DOF asymmetric model could not be obtained.

The number of independent variables is 48 for the symmetric model, and 147 for the asymmetric model. The optimization results of three models are shown in Table 1. All three solutions are regarded to satisfy the conditions of developability and existence of quadrilateral facets with good accuracy. We have confirmed by the large-deformation analysis as shown in Fig. 7 that the generated polyhedra can be continuously developed to a plane without deformation of their facets. When the strain of each frame element is small enough throughout the process of large-deformation analysis, the obtained polyhedron is confirmed to be rigid-foldable. Since the geometric errors of the symmetric models are smaller than the asymmetric model, convergence property strongly depends on the number of variables. It should also be noted that optimization process did not converge if geometric conditions are assigned as constraints.



(a) Symmetric model (b) Asymmetric model

Figure 5. Optimal shape of 9-DOF model

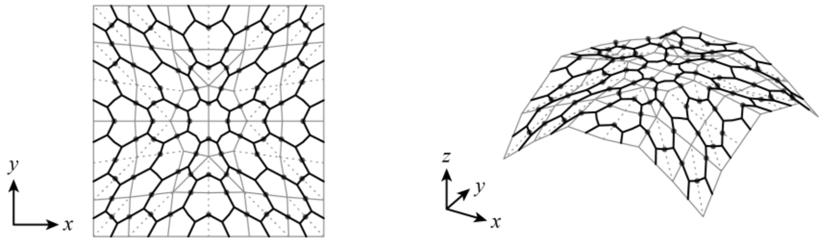
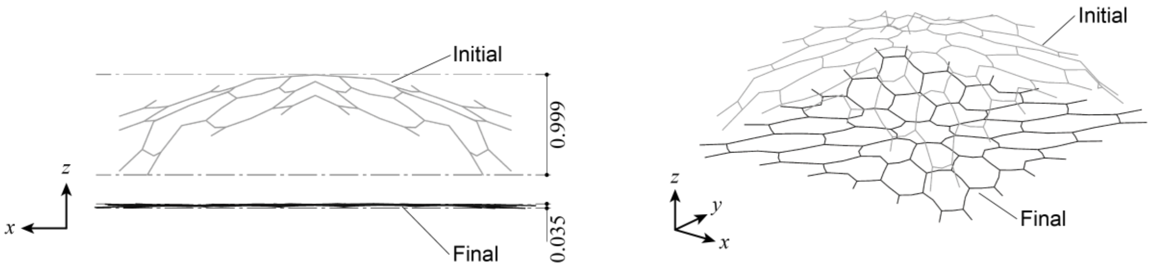


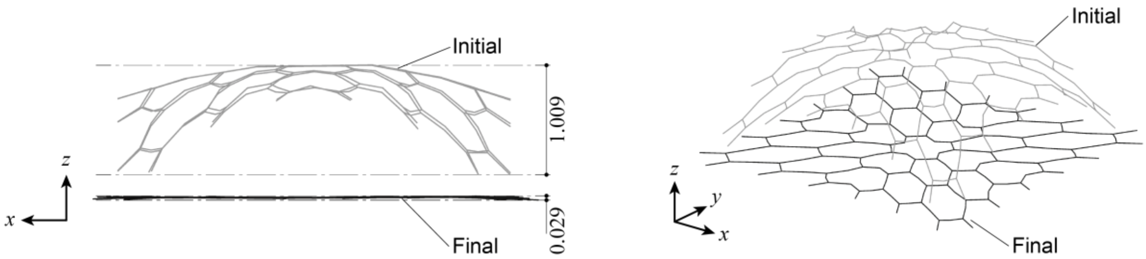
Figure 6. Optimal shape of 5-DOF model

Table 1. Results of optimization

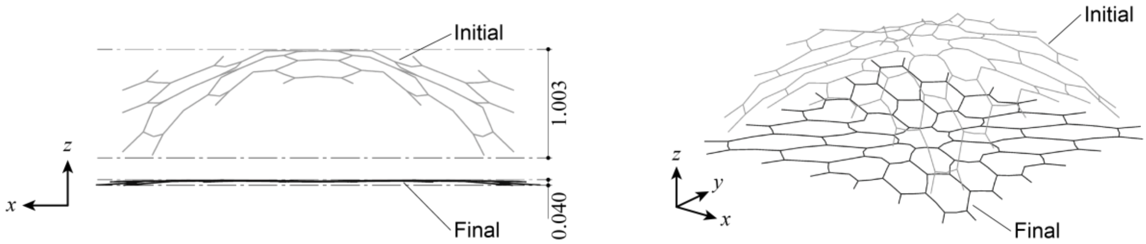
Model	$F(\mathbf{X})$	Max. error of $\sum\theta_{v,k}$ from 360 (deg.)	Max. error of γ_e $(e \in E_D)$ from 180 (deg.)	Max. $\lvert\Delta z_i(\mathbf{X})\rvert$
9-DOF symmetric	2.430×10^{-13}	7.820×10^{-5}	1.017×10^{-5}	2.000
9-DOF asymmetric	1.171×10^{-8}	3.538×10^{-3}	8.680×10^{-4}	0.189
5-DOF symmetric	3.393×10^{-13}	6.505×10^{-5}	5.332×10^{-6}	0.189



(a) 9-DOF symmetric model



(b) 9-DOF asymmetric model



(c) 5-DOF symmetric model

Figure 7. Large-deformation analysis

4. Conclusion

The conclusion of this paper is summarized as follows:

a) The method for form generation of a general rigid-foldable polyhedron in our previous paper [6] has ben extended to incorporate practical situation. The optimization problem is re-formulated to take into account the symmetry condition of the polyhedron. The number of independent variables is reduced using the symmetry condition.

b) We have obtained 6 symmetric optimal shapes whose DOFs range from 1 to 21. Two symmetric models which have 9- and 5-DOFs are selected as examples to be investigated. In addition, 9-DOF asymmetric model is shown for comparison. We have confirmed from large-deformation analysis that each model can be developed to a plane without deformation of its facets.

c) It has been concluded from the examples in Section 3 that the number of variables has a large influence on the efficiency and accuracy of the optimization process. A polyhedron with smaller number of DOF can be found by assigning symmetry condition and fixing the symmetrically located crease lines simultaneously. By contrast, the optimal shape diverts from symmetric shape, if a single crease line is fixed after each process of optimization.

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