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Free-form design of tensegrity − inspired by Snelson's works

Jingyao ZHANG*, Makoto OHSAKI[a]

*Graduate School of Design and Architecture, Nagoya City University, Japan
2-1-10, Kita Chikusa, Chikusa-ku, Nagoya, 464-0083, Japan
zhang@sda.nagoya-cu.ac.jp

[a] Dept. of Architecture and Architectural Engineering, Kyoto University, Japan

Abstract

Inspired by the original artworks by Kenneth Snelson, tensegrity structures have found a great number of applications in many different disciplines. Due to the interplay between member forces and shape, the process of form-finding is the first key step for any application of tensegrity structures.

In this paper, we introduce some of our practices in free-form design of tensegrity structure, including the adaptive force density method, the dynamic relaxation method, the non-linear analysis method, and the optimization method making use of the non-linear analysis method. However, it should be noted that these methods can only serve for designated purposes, and there exists no 'perfect' solution that is applicable to any kind of problems.

The tensegrity tower and tensegrity arch, which were originally proposed by Snelson, are extensively utilized as numerical examples to test validity of the proposed methods.

Keywords: form-finding, tensegrity, force density, dynamic relaxation, non-linear analysis, optimization, shape control.

1. Introduction

The terminology of tensegrity was given by Fuller [1], referring to a prestressed pin-jointed structure consisting of continuous tensile members (cables) and discontinuous compressive members (struts). Tensegrity structures are usually stabilized by the introduction of prestress.

Compared to conventional structural forms such as trusses, there are two distinct characteristics lying in tensegrity structures: (a) they are free-standing without any support; and (b) their cables in tension are continuous, while the struts in compression are discontinuous.

The existence of prestresses in tensegrity structures leads to difficulties in the determination of their (self-equilibrated) configurations associated with prestresses, since every node has to be balanced by the prestresses in the members. Being free-standing is another source of difficulties, because the existing methods for some other tension structures suspended to supports, such as cable-nets, cannot be directly applied to tensegrity structures.

The process of determining the self-equilibrated configuration as well as distribution of prestresses, which is called *form-finding* or *shape-finding*, is one of the key design problems for tensegrity structures. Since the design problem of tensegrity structures attracts attention of academic researchers, there have been a number of form-finding methods developed. Detailed review of these methods can be found in many existing literatures, see for instance [2–4].

The existing methods can be generally categorized as intuition methods, analytical methods, and numerical methods. Due to the high non-linearity of form-finding problems, only numerical methods are generally applicable to complex structures.

In this paper, we summarize some of our previous and current studies on numerical form-finding methods: Section 2 introduces the basic knowledge on self-equilibrium of tensegrity structures; Section 3 presents the adaptive force density method, which guarantees a stable structure [5], and has precise control over height as well as overlap between the unit cell of a tensegrity tower [6]; Section 4 describes dynamic relaxation method, in which the static self-equilibrium equations are solved in dynamics [7]; Section 5 introduces the non-linear analysis method, in which the rigid-body motions are excluded by taking care of the stiffness matrix [8]; Section 6 presents an optimization method incorporation with the non-linear analysis method to have more control over the final shape of a tensegrity arch [9]. Finally, brief discussions are given in Section 7.

These methods proposed by the authors were intentionally developed for finding the shapes of tensegrity tower and tensegrity arch, which were originally proposed by Snelson. They can be further applied to other tensegrity structures with complicated shapes.

2. Self-equilibrium

Suppose that a tensegrity structure has *n* free nodes and *m* members. Denoting prestress (axial force) vector in the members by $\mathbf{s} \in \mathbb{R}^m$, and the resulting forces at the nodes in each direction as $\mathbf{p}^x$, $\mathbf{p}^y$, and $\mathbf{p}^z (\in \mathbb{R}^n)$, the equilibrium equations in each direction can be written as follows

$$\begin{aligned} \mathbf{D}^x\mathbf{s} &= \mathbf{p}^x, \\ \mathbf{D}^y\mathbf{s} &= \mathbf{p}^y, \\ \mathbf{D}^z\mathbf{s} &= \mathbf{p}^z. \end{aligned} \tag{1}$$

In the above equation, $\mathbf{D}^x$, $\mathbf{D}^y$, and $\mathbf{D}^z (\in \mathbb{R}^{n\times m})$ are the equilibrium matrices in three directions.

Denote the force density q_k of member *k* as the force s_k to length l_k ratio; i.e., $q_k = s_k / l_k$. The force density vector is denoted by $\mathbf{q} \in \mathbb{R}^m$. The equilibrium equations can be written into another form with respect to the nodal coordinates

$$\begin{aligned} \mathbf{E}\mathbf{x} &= \mathbf{p}^x, \\ \mathbf{E}\mathbf{y} &= \mathbf{p}^y, \\ \mathbf{E}\mathbf{z} &= \mathbf{p}^z, \end{aligned} \tag{2}$$

where $\mathbf{E} \in \mathbb{R}^{n\times n}$ is called force density matrix, since it is a function of the force density vector $\mathbf{q}$.

When the structure is in the state of self-equilibrium, the resulting forces $\mathbf{p}^x$, $\mathbf{p}^y$, $\mathbf{p}^z$ are zero, such that

$$\mathbf{D}^x\mathbf{s} = \mathbf{D}^y\mathbf{s} = \mathbf{D}^z\mathbf{s} = \mathbf{E}\mathbf{x} = \mathbf{E}\mathbf{y} = \mathbf{E}\mathbf{z} = \mathbf{0}, \tag{3}$$

which are called self-equilibrium equations. The purpose of form-finding of tensegrity structures is to find the force densities **q** or prestress **s** as well as the geometry realization in terms of nodal coordinates **x**, **y**, **z**.

3. Adaptive force density method

The force density method was originally proposed for form-finding of cable nets [10]. It is very effective because only linear equations need to be solved in the form-finding process, and is found to be particularly suitable for searching new configurations.

In the adaptive force density method, the force densities and nodal coordinates are unknown parameters. The process is divided into two subsequent design stages: (a) finding the feasible force densities that satisfy the non-degeneracy condition, and (b) determining the self-equilibrium configuration, in terms of nodal coordinates. The geometrical constraints in linear forms with respect to force densities and nodal coordinates can be respectively incorporated into these two design stages.

Rank deficiency of **E** has to be larger than three for a non-degenerate structure in three-dimensional space. The symmetric force density matrix is decomposed as follows [11]

$$\mathbf{E} = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^{\mathrm{T}}. \tag{4}$$

The diagonal entries of the diagonal matrix $\mathbf{\Lambda}$ are eigenvalues of $\mathbf{E}$, and their corresponding eigenvectors are summarized in $\mathbf{\Phi}$. Let $\mathbf{E}$ have the necessary rank deficiency by artificially modifying four of the diagonal entries of $\mathbf{\Lambda}$ to zero, and reconstruct the updated version $\bar{\mathbf{E}}$ of $\mathbf{E}$ with the new diagonal entries in $\bar{\mathbf{\Lambda}}$ as

$$\bar{\mathbf{E}} = \mathbf{\Phi}\bar{\mathbf{\Lambda}}\mathbf{\Phi}^{\mathrm{T}}. \tag{5}$$

The new force densities are then calculated by using $\bar{\mathbf{E}}$. The force densities can be updated by using the following equation

$$\mathbf{q} = \mathbf{\Psi}(\mathbf{B}\mathbf{\Psi})^{+}\mathbf{g}, \tag{6}$$

in which the linear constraints on force densities, including those from $\bar{\mathbf{E}}$, symmetry, as well as elevation constraints are incorporated into $\mathbf{\Psi}$, $\mathbf{B}$, and $\mathbf{g}$. Moreover, $()^{+}$ denotes the Moore-Penrose generalized inverse of the matrix. Since the new force densities are approximate solutions, the updated force density matrix may not have the required rank deficiency, and iterative calculations are necessary.

Algorithm 1: Adaptive force density method

Step 0: Specify an initial force density vector $\mathbf{q}^0$ to obtain $\mathbf{E}^0$. Formulate $\mathbf{B}^0$ and $\mathbf{g}^0$ with the specified linear constraints. Set i:=0..

Step 1: Set four of the smallest singular values of $\mathbf{E}^i$ to zero to reconstruct $\bar{\mathbf{E}}^i$ by Eq.(5).

Step 2: Obtain $\mathbf{g}^{i+1}$, calculate $\mathbf{q}^{i+1}$ by Eq.(6) and update $\mathbf{E}^{i+1}$..

Step 3: Check the rank of $\mathbf{E}^{i+1}$; i.e. if it has rank deficiency of four, then let $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$, construct $\hat{\mathbf{E}}$ from $\hat{\mathbf{q}}$, and terminate the algorithm; otherwise, set $i \leftarrow i+1$ and return to Step 1.

As the force density matrix in terms of force densities has been determined, the configuration in terms of nodal coordinates can then be determined by satisfying the self-equilibrium equations associated with the force density matrix and the nodal coordinates.

As a numerical example, we consider the ten-layer tensegrity tower as shown in Figure 1. It is assembled by the unit cell with four struts. Rotational symmetry about z-axis is imposed, and the unit cells are assigned to have the same height as well as the same overlaps between the units. Although we can have precise control over the elevation of the structure, it is not easy to control its shape in the xy-plane, as can be observed from its final shape in Figure 1.

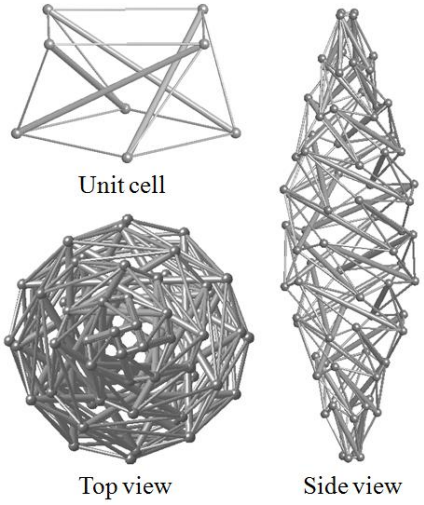


Figure 1: Self-equilibrium configuration of the ten-layer tensegrity tower by force density method.

4. Dynamic relaxation method

Dynamic relaxation method (DRM) is an explicit solution technique, originally developed for tidal flow computations. It has been successfully applied to the form-finding problems of cable nets and membrane structures [12], and then to the problem of non-regular tensegrity structures [13].

The form-finding process in DRM starts from an initially specified configuration as well as prestresses. Because of the existence of prestresses, the initial design of the structure is usually not in the equilibrium state, with out-of-balance forces at each node. Therefore, motion occurs at each node, which is traced at every step of small time increments in dynamic relaxation method.

When the total kinetic energy of the system arrives at a local peak, all velocity components are set to zero, and the process is restarted from the new configuration and prestresses. To accelerate convergence of the process, fictitious nodal mass components and viscous damping are artificially assigned. The motion tracing and restarting procedures are continued until the latest local peak is sufficiently small, at which the structure is considered to be in the self-equilibrium state.

Algorithm 2: Dynamic relaxation method

Step 0: Set the starting time as t:=0. Specify the initial configuration with the generalized coordinates $\mathbf{X}_0$ and prestresses $\mathbf{s}_0$. Specify time interval Δt for pseudo time history analysis with zero initial velocities $\mathbf{v}_0=\mathbf{0}$.

Step 1: Derive the nodal accelerations $\mathbf{a}_t$ at time t as

$$\mathbf{a}_t = -\mathbf{M}^{-1}(\mathbf{C}\mathbf{v}_t + \mathbf{f}_t) \ , \tag{7}$$

where $\mathbf{M}$ and $\mathbf{C}$ are the matrices for mass and damping, respectively; and $\mathbf{f}$ is the vector for residual forces. The nodal velocities and coordinates at time $t+\Delta t$ are updated as

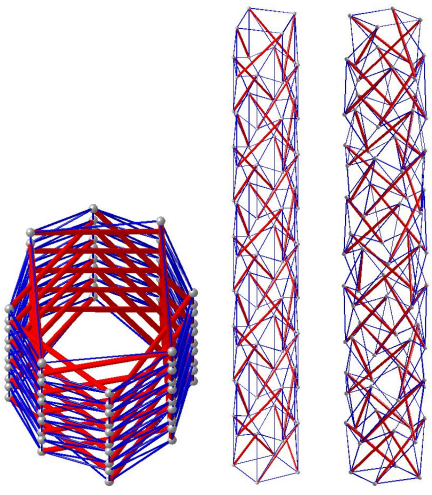
$$\mathbf{v}_{t+\Delta t} = \mathbf{v}_t + \mathbf{a}_t \Delta t \ ,$$

$$\mathbf{X}_{t+\Delta t} = \mathbf{X}_t + \mathbf{v}_t \Delta t \ . \tag{8}$$

Step 2: If the mean square of velocities reaches its local peak, then reset the velocities to zero.

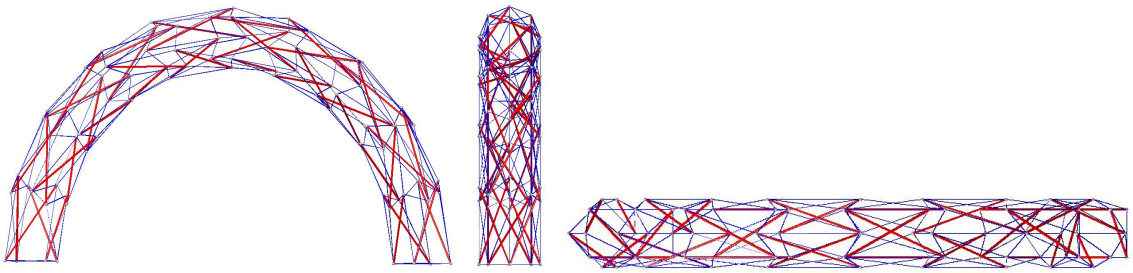
Step 3: If the mean residual force is smaller than the specified small value, then terminate the algorithm; otherwise, return to Step 1 with the time updated as t=:$t+\Delta t$.

As a numerical example, we consider a tensegrity tower and a tensegrity arch, with the initial configurations in Figures 2(a),(b) and 3. The self-equilibrated configurations are shown in Figures 2(c) and 4. It can be observed that the final configurations are very close to their initially specified ones.



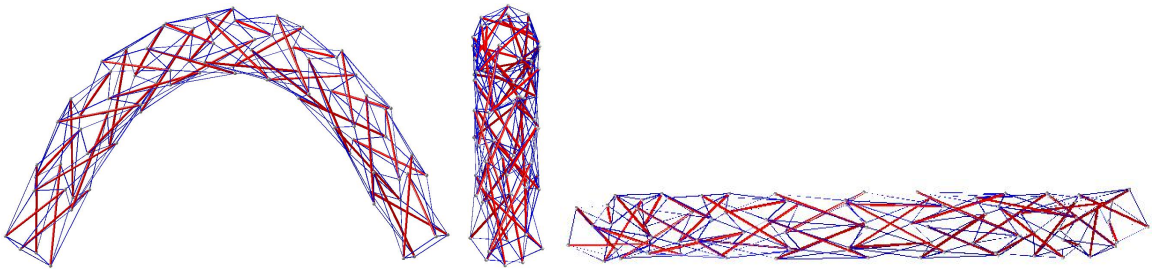
(a) Initial (b) Initial (c) Final

Figure 2. Ten-story tensegrity tower by dynamic relaxation method.



(a) Diagonal view (b) Side view (c) Top view

Figure 3. Initial configuration of the tensegrity arch.



(a) Diagonal view (b) Side view (c) Top view

Figure 4. Self-equilibrated tensegrity arch by dynamic relaxation method.

5. Non-linear analysis method

In this section, we present the non-linear analysis for finding self-equilibrated configurations of tensegrity structures. Since a tensegrity structure is free-standing without any support to constrain its rigid-body motions, its tangent stiffness matrix is non-invertible subsequently. To conduct the non-linear analysis, the generalized inverse of the tangent stiffness matrix is computed by making use of singular value decomposition.

The nodal displacements, denoted by the displacement vector $\mathbf{d} \in \mathbb{R}^{3n}$, are calculated as follows by using the generalized inverse of $\mathbf{K}$, while the structure is subjected to external loads $\mathbf{f} \in \mathbb{R}^{3n}$:

$$\mathbf{d} = \mathbf{K}^{+}\mathbf{f} \tag{9}$$

The tangent stiffness matrix $\mathbf{K}$ is decomposed as follows by using spectral decomposition

$$\mathbf{K} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{\mathrm{T}}, \tag{10}$$

where $\mathbf{\Lambda}$ is a diagonal matrix, the diagonal entries Λ_{ii} of which are the singular values of $\mathbf{K}$. To rule out the rigid-body motions, without imposition of additional displacement constraints, the Moore-Penrose generalized inverse $\mathbf{K}^{+}$ of $\mathbf{K}$ is calculated by substituting the following equation for Λ_{ii}^{-1}

$$\Lambda_{ii}^{-1} = \begin{cases} 1/\Lambda_{ii} & \text{if } \Lambda_{ii} \neq 0, \\ 0 & \text{if } \Lambda_{ii} = 0. \end{cases} \tag{11}$$

Because the rigid-body motions corresponding to the zero eigenvalues of $\mathbf{K}$, application of Equation (11) means that these motions are locked, resulting in no displacements along the directions of these motions. Thus, there is no need to constrain the rigid-body motions *a priori* to compute the displacements.

Algorithm 3: Non-linear analysis method

Step 0: Specify initial values for nodal coordinates **x**, **y**, **z**, and prestresses **s**.

Step 1: Calculate the unbalanced forces $\mathbf{f}^{x}, \mathbf{f}^{y}, \mathbf{f}^{z}$ in each direction. The algorithm terminates if the unbalanced forces are sufficiently small; otherwise, continue to Step 2.

Step 2: Construct the tangent stiffness matrix **K**, calculate the generalized inverse $\mathbf{K}^{+}$, and then derive the displacements **d** subjected to the unbalance forces $\mathbf{f}^{x}, \mathbf{f}^{y}, \mathbf{f}^{z}$.

Step 3: Update the prestresses **s** and the configuration in terms of nodal coordinates. Return back to Step 1.

The non-linear analysis method essentially terminates at the same final self-equilibrated configuration as that derived by DRM, if the same initial settings are assigned.

6. Shape optimization method

In this section, we combine the non-linear analysis method in the previous section together with the optimization technique to have more control over the final configuration satisfying designers' requirements on geometry.

There are two levels of processes in the proposed method: the *global* process is formulated as an optimization problem, which is to find the configuration as close as possible to the expected one; and the *local* process determines the self-equilibrated configuration of the structure by using the non-linear analysis, with the assigned parameters in the global process.

In our previous study, we have demonstrated that the self-equilibrated configuration of a tensegrity tower can be controlled by using different stiffnesses for the members [14]. Hence, we use the member stiffnesses $(AE)_k$ as variables to find the final configuration of the structure that is close enough to the expected one.

Consider, for example, the form-finding problem of a curved tensegrity tower, which is a bent version of the tensegrity towers. The objective function can be defined as distance *d* from the center point of top or bottom circle of each layer to the targeted curve. Therefore, the optimization problem is formulated as

$$\begin{aligned} &\text{Minimize} && d \\ &\text{subjected to} && k^{\mathrm{L}} \le (EA)_k \le k^{\mathrm{U}}, \end{aligned} \tag{12}$$

where k^{L} and k^{U} are respectively lower and upper bounds of $(EA)_k$. The global process of the form-finding approach is summarized as follows:

Algorithm 4: Shape optimization method:

Step 0: Specify the initial values for the nodal coordinates **x**, **y**, **z**, the prestresses **s**, and the stiffnesses $(AE)_k$ of a tensegrity tower.

Step 1: Find the self-equilibrated configuration of the structure with the specified parameters by the application of Algorithm 3, and compute the objective function *d*.

Step 2: Use optimization technique to search for new stiffnesses $(EA)_k$ leading to smaller objective function.

Step 3: If the termination condition is not satisfied, then go to Step 1.

The proposed method is applied to find a curved tensegrity structure making use of the tensegrity tower. Top view and side view of its initial configuration are shown in Figures 5(a1) and (a2), respectively. It is constructed by assembling ten unit cells, each of which consists of three struts. The unit cells are connected by diagonal cables as indicated by green lines in Figure.

The stiffnesses of the struts and cables are assigned as 100.0 and 10.0, respectively. The initial presstresses in the struts and cables are assigned as −2.0 and 2.0, respectively. Units in the example are omitted, since they have no effect on the self-equilibrated configuration.

Using the initially given stiffnesses, the self-equilibrated configuration of the tensegrity tower is shown in Figures 5(b1) and (b2). The mean distance is 2.0459.

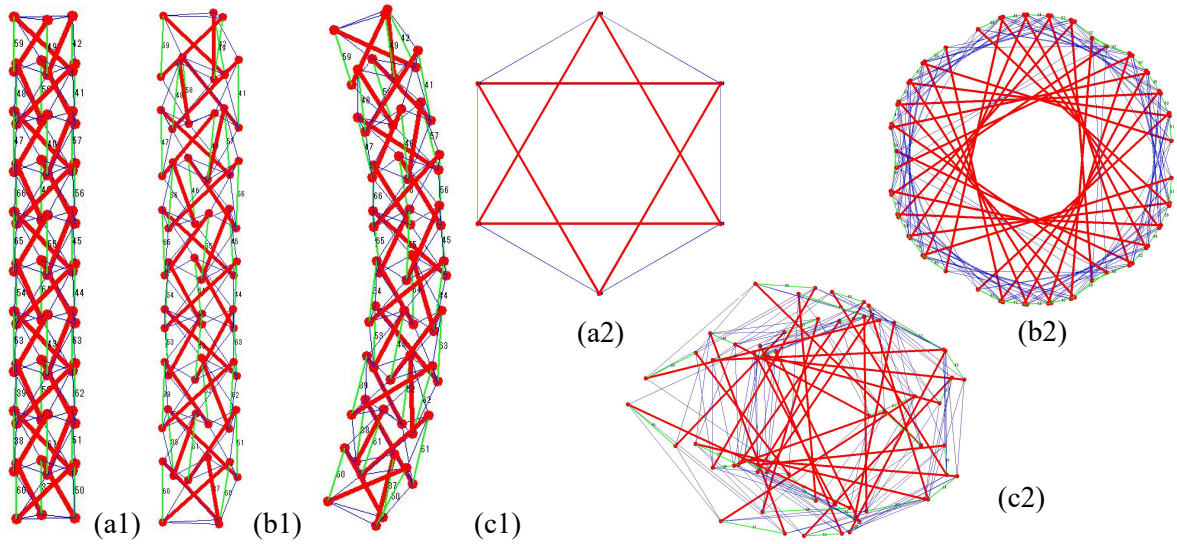


Figure 5. Configurations of the curved tensegrity structure.

The diagonal cables are classified into six groups, and their stiffnesses are variables in the form-finding process. The upper bound and lower bound of their stiffnesses are set as 100.0 and 0.01, respectively. The targeted curve is an arc with the opening angle $\pi/4$. Sequential quadratic programming (SQP) is adopted to solve the optimization problem, and it returns the final configuration in Figures 5(a3) and (b3). The mean distance in the final solution is reduced to 0.5591.

7. Conclusions

For the problem of determining self-equilibrium configurations for tensegrity structures, we have summarized four of our numerical methods in this paper. Each method has its own merits as well as demerits, and no single method can cater for all requirements by the designers:

(1) The adaptive force density method has great advantages in guaranteeing a (super-)stable structure, and also has some controls over geometry of the structures; however, it is still weak in free-form design as also can be observed in the numerical example.

(2) The dynamic relaxation method can find a self-equilibrated configuration close to that is initially specified by the designers. However, it suffers from low convergence performance, which needs further studies. To speed up the convergence, parameter tuning for relative simple structures, for instance unit cell, could be helpful.

(3) The basic concept of non-linear analysis method is much more familiar to structural engineers. Its convergence performance is much better than the dynamic relaxation method, and it may have the problem in dealing with complicated structures consisting of a large number of nodes and members. This comes from the fact that eigenvalue analysis of the tangent stiffness matrix is necessary in each iteration of the analysis.

(4) To enable designers have more control over the self-equilibrated configurations in a systematic way, the non-linear analysis method is incorporated into an optimization problem. The global process of the method solving the optimization problem is to find the closest configuration to the targeted one, and the local process solving a non-linear analysis problem is to find the self-equilibrated configuration of the structure with the parameter values given by the global process. More variables will result in more accurate shape control, however, may lead to slower convergence.

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References

[1] R.B. Fuller, *Synergetics, Explorations in the Geometry of Thinking*, Collier Macmillan, London, 1975.

[2] A. G. Tibert and S. Pellegrino, "Review of form-finding methods for tensegrity structures," *International Journal of Space Structures,* vol. 26, no. 3, pp. 241-255, 2011.

[3] C. Sultan, "Tensegrity: 60 years of art, science, and engineering", *Advances in applied mechanics* (Book Series: Advances in Applied Mechanics), Vol. 43, pp. 69-145, 2009.

[4] S.H. Juan and J.M.M. Tur, "Tensegrity frameworks: static analysis review", *Mechanism and Machine Theory*, Vol., No.7, pp. 859-881, 2008.

[5] J.Y. Zhang and M. Ohsaki, "Adaptive force density method for form-finding problem of tensegrity structures", *Int. J. Solids, Struct.*, Vol. 43, pp. 5658-5673, 2006.

[6] J.Y. Zhang and M. Ohsaki, "Form-finding of tensegrity structures subjected to geometrical constraints", *Int. J. Space Structures*, Vol. 21, 4, pp. 183-195, 2006.

[7] J.Y. Zhang and M. Ohsaki, "Form-finding of complex tensegrity structures by dynamic relaxation method", *J. Struct. Constr. Eng.*, AIJ, Vol. 81, No. 719, pp. 71-77, 2016.

[8] J.Y. Zhang and M. Ohsaki, "Free-form design of tensegrity structures by non-rigid-body motion analysis", *Proc. International Association for Shell and Spatial Structures*, Wroclaw, Poland, Sep. 2013.

[9] J.Y. Zhang and M. Ohsaki, "Free-form design of tensegrity structures by combination of non-linear analysis and optimization techniques", *Spatial Structures*, Vol. 29, No. 3, pp. 92-96, 2015.

[10] K. Linkwitz and H.-J. Schek, "Einige bemerkungen zur berechnung von vorgespannten seilnetzkonstruktionen", *Ingenieur-Archiv*., Vol. 40, 1971, pp. 145–158.

[11] G.H. Golub and C.F. van Van Loan, *Matrix Computations* (3rd Edition), Johns Hopkins Studies in Mathematical Sciences, 1996.

[12] M.R. Barnes, "Form-finding and analysis of prestressed nets and membranes", *Comput. Struct.*, Vol. 30, 1988, pp. 685–695.

[13] L. Zhang, B. Maurin and R. Motro, "Form-finding of nonregular tensegrity systems", *J. Structural Engineering, ASCE*, Vo. 132, No. 9, 2007, pp. 1435-1441.

[14] M. Ohsaki and J.Y. Zhang, "Nonlinear programming approach to form-finding and folding analysis of tensegrity structures using fictitious material properties", *International Journal of Solids and Structures*, Vol 69-70, pp. 1-10, 2015.