# Series expansion of regular 2 N -gonal frame with inclined joints foldable into straight rod shape 

Ryo WATADA*, Makoto OHSAKI ${ }^{a}$<br>* Department of Architecture and Architectural Engineering, Kyoto University<br>Kyoto-Daigaku Katsura, Nishikyo, Kyoto 615-8540, Japan<br>se.watada@archi.kyoto-u.ac.jp<br>${ }^{\text {a }}$ Department of Architecture and Architectural Engineering, Kyoto University, Japan


#### Abstract

A regular $2 N$-gonal frame which has $2 N$ bars connected with two types of inclined revolute joints and is foldable into entirely straight rod shape is presented. This frame has an infinitesimal mechanism of at least fourth order, which is a necessary condition for the frame having a finite mechanism. Incremental geometrically nonlinear analysis shows that the path of the mechanism leads to a state where all bars of the frame are completely folded into a rod if a cross-section of all bars is set to be an isosceles triangle whose vertex angle is $\pi / N$ and the axes of the bar-hinges are directed to the intersection of inner or outer lateral faces of two adjacent bars. As an application, a dome-shaped deployable structure that can be entirely folded into straight rod shape is proposed.


Keywords: series expansion method, regular 2 N -gonal frame, inclined joints, foldable structure.

## 1. Introduction

Linkage mechanism is defined as a structure that can have deformation without external loads. A mechanism is said to be an infinitesimal mechanism if deformation without force is allowed only if the deformation is small. By contrast, a mechanism is called finite mechanism if it can have large deformation without force, which is more important for practical application. Furthermore, infinitesimal mechanism has order [1] and finite mechanism is represented as the case where the order is infinity. Therefore, evaluation of the order of infinitesimal mechanism is important to investigate various properties of linkage mechanisms. Ohsaki et al. [2] proposed an optimization-based approach solving linear programming problem for obtaining bar-hinge frames which have infinitesimal mechanism. Watada et al. [3] presented a method to calculate the order of infinitesimal mechanism of bar-hinge structures using series expansion of compatibility conditions at revolute joints.
In this study, we present a regular $2 N$-gonal frame which has $2 N$ bars connected with inclined revolute joints. The bars have the same length, and the $2 N$ hinges are classified into two types. Accordingly, the structure has a dihedral symmetry. It is shown through the series expansion method [3] that the frame has a mechanism of at least fourth order, which is a necessary condition for the frame having a finite mechanism. Furthermore, assuming that a cross-section of all bars is set to be an isosceles triangle whose vertex angle is $\pi / N$ and the axes of the bar-hinges are directed to the intersection of inner or outer lateral faces of two adjacent bars, we show that the path of the mechanism leads to a state where all bars of the frame are completely folded into a rod by incremental geometrically nonlinear analysis. It should be remarked that the structure is known as Bennet linkage [4] and threefold-symmetric Bricard linkages [5] when $2 N$ is equal to 4 and 6 , respectively. Here, we consider the case where $2 N$ has any even number greater than or equal to 8 . Especially, the case $2 N$ is equal to 12 is studied in detail. Finally, combining the regular polygonal frames, we generate a dome-shaped deployable structure that can be entirely folded into straight rod shape.

## 2. Regular 2 N -gonal frame with isosceles triangle cross-section

### 2.1. Concept

Consider a regular 2 N -gonal frame which has 2 N bars connected with two types of inclined revolute joints as shown in Fig. 1(a). All bars have an isosceles triangle whose vertex angle is $\theta=\pi / N$ as shown in "A-A section" in Fig.1(a). The axes of the bar-hinges are directed to the intersection of inner or outer lateral faces of two adjacent bars shown in Fig.1(a) by dashed lines with red and blue colors. Then, the elevation angle $\beta$ of these hinges shown in "B-B" and "C-C" sections are calculated as

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\begin{equation*}
\beta=\arctan \left\{\frac{1}{\tan \left(\frac{\pi}{2 N}\right) \sqrt{1+\left\{\tan \left(\frac{\pi}{2 N}\right)\right\}^{2}}}\right\} \tag{1}
\end{equation*}
$$

For numerical analysis, this frame is simply modeled with two rigid bars and four inclined hinges as shown in Fig. 1(b), where $1 / N$ of the frame is modeled because of symmetry. In Fig.1(b), $\boldsymbol{l}_{k}^{1}, \boldsymbol{l}_{k}^{2}$ and $\boldsymbol{l}_{\boldsymbol{k}}^{3}$ denote the local coordinate vectors of node $k$ where $\boldsymbol{l}_{k}^{3}$ is in the direction of $z$-axis, and the notation [T1, T2, T3, R1, R2, R3] represent support conditions of translation and rotation; the first characters ' T ' and ' $R$ ' indicate that displacement and rotation, and the second characters ' 1 ', ' 2 ' and ' 3 ' represent the number of local coordinates for the direction or axis of rotation.


Figure 1: Regular 2 N -gonal frame with an isosceles triangle cross-section; (a) Overall model and details of joints, (b) Simplified $1 / N$ model.

First, we consider the case $2 N=12$ and calculate the order of mechanism in Fig.1(b) using series expansion method [3]. In Fig. 1(b), the numbers of members $m_{0}$, nodes $n_{0}$, hinges $h$ and constraints $\mathbf{c}$ are 2, 3, 4 and 10 , respectively. Accordingly, the number $n$ of components of the generalized displacement vector $\boldsymbol{W}$ is determined as $n=6 n_{0}+6 m_{0}=30$, and the number $m$ of components of the incompatibility vector $\boldsymbol{G}(\boldsymbol{W})$ is $m=12 m_{0}-h+c=30$. The hinge angle $\beta$ is determined as 1.300 rad from Eq. (1), and the compatibility matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{m \times n}$ is defined as $\boldsymbol{\Gamma}=\left[\partial G_{i} /\left.\partial W_{j}\right|_{W=0}\right]$. From the singular value decomposition (SVD), the rank $u$ of the matrix $\boldsymbol{\Gamma}$ is obtained as $u=\operatorname{rank}(\boldsymbol{\Gamma})=29$. Therefore, the number of infinitesimal mechanism modes $p$ and the number of self-equilibrium modes $q$ are determined as $p=n-u=1$ and $q=m-u=1$, respectively. Using series expansion method, it is confirmed that this model satisfies the conditions for existence of at least forth order infinitesimal mechanism with respect to the path parameter, which predicts that this model has a finite mechanism, and the mechanism modes including at most fourth order terms were obtained. Fig. 2 shows the variation of the vertical displacement $U_{V}$ of node $\mathrm{P}_{1}$ with respect to the horizontal displacement $U_{H}$ of the same node, where 'SeriesExp1stOrder' considers only first order term of mechanism whereas 'SeriesExp2ndOrder', ‘SeriesExp3rdOrder' and 'SeriesExp4thOrder' include up to the second, third and fourth order terms, respectively.

Next, an incremental large deformation analysis is carried out to confirm that the frame has a finite mechanism and to obtain the deformation path numerically. For the large deformation analysis, we used
the whole model shown in Fig. 3 which has all 12 members, 12 nodes $\mathrm{P}_{0}, \ldots, \mathrm{P}_{11}, 24$ hinges and constraints as an assemblage of the 6 rotated copies of Fig. 1(b). As a path parameter, forced deformation in negative $\boldsymbol{l}_{1}^{1}$ direction is applied at node $\mathrm{P}_{1}$. Abaqus Ver. 6.14 is used for the analysis.
The obtained mechanism is shown in Fig. 3. It should be remarked that the path of the mechanism shown in Fig. 3 leads to a state where all bars of the frame are completely folded into a rod. The variation of the displacement on node $P_{1}$ calculated by the incremental analysis is indicated with dotted line in Fig. 2, which converges to -0.5176 : the length of the sides of the frame. It can be seen from Fig. 3 that lines obtained by the series expansion, especially 'SeriesExp4thOrder', has good accuracy compared with dotted line obtained by incremental analysis. The physical prototype of this frame is shown in Fig. 4.


Figure 2: Variation of the vertical displacement with respect to the horizontal displacement on node $\mathrm{P}_{1}$.


Figure 3: Deformation path of regular 12-gonal frame obtained by incremental large deformation analysis.


Figure 4: Physical prototype of regular 12-gonal frame.

## 3. Application

To confirm that the generalized $2 N$-gonal frames also have the same property, we examined the cases $2 N=8,10$ and 16 as examples using the series expansion method as described above, and confirmed that these frames satisfy the necessary conditions for existence of finite mechanism at least third order.
Actually, incremental large deformation analysis shows that these frames also have deformation paths leading to the states where all bars of the frames are folded into rods. Furthermore, for every $2 N$ from 4
to 20 , the deformation paths with the same property are obtained as shown in Fig. 4. It should be noted that the cases for $2 N=4$ and 6 are known as Bennet linkage [4] and Bricard linkage [5], respectively, although these two linkages both have one finite mechanism except for rigid body deformations even if no constrains are given.

Finally, as an application, we propose a dome-shaped deployable structure by combining the regular polygonal frames as layers in vertical direction as shown in Fig. 5(a). Two vertically adjacent nodes are connected with horizontal hinges as shown in Fig. 5(b). The proposed dome can be entirely folded into straight rod shape as shown in Fig. 5(c).


Figure 5: Mechanisms of the regular $2 N$-gonal frames ( $4 \leq 2 N \leq 20$ ).


Figure 6: Dome-shaped deployable structure as an application of the regular 2 N -gonal frames; (a) Combination of the frames, (b) Connection between two vertically adjacent nodes, (c) Deformation path.

## 4. Conclusion

The main conclusions are summarized as follows.

1) A regular 2 N -gonal frame which has 2 N bars with an isosceles triangle cross-section connected with inclined hinge joints is presented. It is shown by the series expansion method and the incremental largedeformation analysis that the proposed frame is foldable into entirely straight rod shape without strain.
2) A dome-shaped structure foldable into straight rod is proposed by combining the regular polygonal frames as layers in vertical direction.
3) It is shown through the regular $2 N$-gonal frame that we can calculate the nonlinear mechanism path with high accuracy by the series expansion method without incremental analysis.

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