

# Approximate method for cutting pattern optimization of membrane structures

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# Abstract

A computationally efficient method is presented for approximate optimization of cutting pattern of membrane structures. The plane cutting sheet is generated by minimizing the error from the shape obtained by reducing the stress from the desired curved shape. The equilibrium shape is obtained solving a minimization problem of total strain energy. The external work done by the pressure is also incorporated for analysis of pneumatic membrane. An approximate method is also proposed for analysis of an Ethylene TetraFluoroEthylene (ETFE) film, where elasto-plastic behavior is modeled as a nonlinear elastic material under monotonic loading condition. Efficiency of the proposed method is demonstrated through examples of a frame-supported PolyVinyl Chloride (PVC) membrane structure and an air pressured square ETFE film.

Keywords: Membrane structure, cutting pattern optimization, energy minimization, PVC, ETFE, pneumatic membrane

# 1. Introduction

In the process of designing a membrane structure, it is important to achieve a moderately uniform stress distribution to prevent fracture and slackening. Difficulty arises from the fact that the curved surface is generated by connecting plane sheets (Ohsaki and Fujiwara [1]). Although there are many methods for cutting pattern optimization, most of the methods should carry out finite element analysis many times (Ohsaki and Uetani [2], Bletzinger et al. [3]).

For shape design of membrane structures, the material is usually supposed to have orthotropic or isotropic elastic behavior. Therefore, the equilibrium shape analysis for specified cutting pattern can be formulated as a forced displacement problem, which can be solved by minimization of the total strain energy. However, Ethylene TetraFluoroEthylene (ETFE) film has an elasto-plastic property; therefore, it is difficult to optimize the shape using a gradient-based optimization algorithm.

In this study, we present a computationally efficient iterative method for approximate optimization of cutting pattern of membrane structures. The plane cutting sheet is generated by minimizing the error from the shape obtained by reducing the stress from the desired curved shape, which is discretized into triangular finite elements. The equilibrium shape corresponding to the specified cutting pattern is obtained by energy minimization. The external work done by the air pressure is also incorporated for analysis of pneumatic membrane structures. The proposed method is extended to design of ETFE film. Efficiency of the proposed method is demonstrated through examples of a frame-supported PolyVinyl Chloride (PVC) membrane structure and an air-pressured square ETFE film.

# 2. Energy minimization for equilibrium shape analysis

Consider a curved membrane structure discretized by triangular finite elements with constant stress in plane stress state. Let  $\mathbf{D} \in \mathbb{R}^{3\times 3}$  denote the constitutive matrix defining the isotropic or orthotropic

elastic material property of the membrane. In the following, stress is evaluated as the force per unit length of section. Relation between the strain vector  $\mathbf{\varepsilon}_k = (\varepsilon_{k1}, \varepsilon_{k2}, \gamma_k)^T$  and the stress vector  $\mathbf{\sigma}_k = (\sigma_{k1}, \sigma_{k2}, \tau_k)^T$  of the *k*th element is written as

$$\boldsymbol{\sigma}_k = \mathbf{D}\boldsymbol{\varepsilon}_k \tag{1}$$

where the subscripts 1 and 2 in stress and strain components indicate the values in two principal directions.



Figure 1:Local coordinates, node numbers, principal directions, and deformation of triangular element *k*.

Let  $\mathbf{u}_k = (u_k^2, u_k^3, v_k^3)^T$  denote the relative displacements of nodes, as shown in Fig. 1, in local (x,y)coordinates of element k, and  $(x_p, y_p)$  defines the principal directions. The strain-displacement
relation is written using matrix  $\mathbf{C} \in \mathbb{R}^{3\times 3}$  as

$$\mathbf{\varepsilon}_k = \mathbf{C}\mathbf{u}_k \tag{2}$$

The vecor consisting of the global (X, Y)-coordinates of the nodes of cutting sheet is denoted by  $\mathbf{X} \in \mathbb{R}^{2n}$ , where *n* is the number of nodes. The process of finding the equilibrium shape is regarded as a forced displacement problem of the sheet to the specified boundary of the curved surface. Therefore, the strain energy  $S(\mathbf{X})$  of the membrane is regarded as a function of  $\mathbf{X}$  as

$$S(\mathbf{X}) = \frac{1}{2} \sum_{k=1}^{m} A_k \boldsymbol{\varepsilon}_k (\mathbf{X})^T \mathbf{D} \boldsymbol{\varepsilon}_k (\mathbf{X})$$
(3)

where *m* is the number of elements, and  $A_k(\mathbf{X})$  is the area of the *k*th element. The equilibrium shape is obtained by minimizing  $S(\mathbf{X})$  under an appropriate boundary conditions.

If air pressure p is given, the pressure potential energy  $W(\mathbf{X})$  is given as (Bouzidi and Le van [4], Fischer [5])

$$W(\mathbf{X}) = pV(\mathbf{X}) \tag{4}$$

where  $V(\mathbf{X})$  is the volume of membrane structure. Using the divergence theorem, Bouzidi and Le van [4] derived the following expression  $W^*(\mathbf{X})$  for a membrane discretized by triangular finite elements:

$$W^{*}(\mathbf{X}) = \frac{p}{3} \sum_{k=1}^{m} A_{k} \mathbf{n}_{k} (\mathbf{X})^{T} \mathbf{X}_{k}^{0} (\mathbf{X})$$
(5)

where  $\mathbf{n}_k(\mathbf{X}) \in \mathbb{R}^3$  and  $\mathbf{X}_k^0(\mathbf{X}) \in \mathbb{R}^3$  are the unit normal vector and coordinate vector of the center of gravity of the *k*th element. However, the term 1/3 in the right-hand-side of Eq. (5) is not necessary, because we should consider shape variation only in the normal direction of surface. Actually they did not use 1/3 in the numerical examples.

Let  $\mathbf{X}_i \in \mathbb{R}^3$  denote the coordinate vector of node *i*. Variation of  $W(\mathbf{X})$  is directly computed as

$$\delta W(\mathbf{X}) = p \delta V(\mathbf{X})$$

$$= p \sum_{k=1}^{m} \sum_{i \in I_{k}} \frac{\partial V(\mathbf{X})}{\partial \mathbf{X}_{i}} \delta \mathbf{X}_{i}$$

$$= p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \mathbf{n}_{k} (\mathbf{X})^{T} \frac{\partial \mathbf{X}_{k}^{0} (\mathbf{X})}{\partial \mathbf{X}_{i}} \delta \mathbf{X}_{i} \right)$$

$$= p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( \frac{A_{k}}{3} \mathbf{n}_{k} (\mathbf{X})^{T} \delta \mathbf{X}_{i} \right)$$
(6)

where  $I_k$  is the set of nodes of element k, and

$$\mathbf{X}_{k}^{0} = \frac{1}{3} \sum_{i \in I_{k}} \mathbf{X}_{i}$$
<sup>(7)</sup>

has been used. The equilibrium shape is obtained by minimizing the total potential energy  $\Pi(\mathbf{X})$  defined as

$$\Pi(\mathbf{X}) = S(\mathbf{X}) - W(\mathbf{X}) \tag{8}$$

Differentiation of  $\Pi(\mathbf{X})$  with respect to  $\mathbf{X}_i$  leads to

$$\frac{\partial \Pi(\mathbf{X})}{\partial \mathbf{X}_{i}} = \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \boldsymbol{\varepsilon}_{k} (\mathbf{X})^{T} \mathbf{D} \frac{\partial \boldsymbol{\varepsilon}_{k} (\mathbf{X})}{\partial \mathbf{X}_{i}} \right) - p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \mathbf{n}_{k} (\mathbf{X})^{T} \frac{\partial \mathbf{X}_{k}^{0} (\mathbf{X})}{\partial \mathbf{X}_{i}} \right) - p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \frac{\partial \mathbf{n}_{k} (\mathbf{X})^{T}}{\partial \mathbf{X}_{i}} \mathbf{X}_{k}^{0} (\mathbf{X}) \right)$$

$$= \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \boldsymbol{\varepsilon}_{k}^{T} (\mathbf{X}) \mathbf{D} \frac{\partial \boldsymbol{\varepsilon}_{k} (\mathbf{X})}{\partial \mathbf{X}_{i}} \right) - \frac{p}{3} \sum_{k=1}^{m} A_{k} \mathbf{n}_{k} (\mathbf{X}) - p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \frac{\partial \mathbf{n}_{k} (\mathbf{X})^{T}}{\partial \mathbf{X}_{i}} \mathbf{X}_{k}^{0} (\mathbf{X}) \right)$$

$$(9)$$

By contrast, the equilibrium equation is given as

$$\sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \boldsymbol{\varepsilon}_{k} (\mathbf{X})^{T} \mathbf{D} \frac{\partial \boldsymbol{\varepsilon}_{k} (\mathbf{X})}{\partial \mathbf{X}_{i}} \right) - \frac{p}{3} \sum_{k=1}^{m} A_{k} \mathbf{n}_{k} (\mathbf{X}) = \mathbf{0}$$
(10)

Therefore, the third term, denoted by  $\mathbf{e}$ , in the right-hand-side of Eq. (9) remains as an additional term, which is rewritten as

$$e = -p \sum_{k=1}^{m} \sum_{i \in I_{k}} \left( A_{k} \frac{\partial \mathbf{n}_{k}(\mathbf{X})^{T}}{\partial \mathbf{X}_{i}} \mathbf{X}_{k}^{0}(\mathbf{X}) \right)$$
  
$$= -p \sum_{i=1}^{n} \sum_{k \in K_{i}} \left( A_{k} \frac{\partial \mathbf{n}_{k}(\mathbf{X})^{T}}{\partial \mathbf{X}_{i}} \mathbf{X}_{k}^{0}(\mathbf{X}) \right)$$
(10)

where  $K_i$  is the set of elements connected to node *i*. Although details are omitted, the absolute values of components of **e** are sufficiently small, if the surface is discretized into sufficiently many elements.

## 3. Approximate optimization of cutting pattern

We propose a simple update rule of the stress parameters called reduction stress for approximate cutting pattern optimization. The method is based on the inverse process of generating a plane sheet from a curved surface by reducing the stress (Ohsaki and Fujiwara [1]). The algorithm is illustrated in Fig. 2, and summarized as follows:



Figure 2: Scheme of approximate cutting pattern optimization.

- Step 1: Assign the target equilibrium shape, boundary condition, target stress, and generate triangular meshes on the target surface. Compute the edge lengths  $L_{k1}$ ,  $L_{k2}$ , and  $L_{k3}$  of the *k*th triangle of the target equilibrium shape. Specify the ideal target stresses  $\sigma_{k1}^*$ ,  $\sigma_{k2}^*$ , and  $\tau_k^*$  (=0), and initialize the step counter s = 0.
- Step 2: Specify the reduction stresses  $\hat{\sigma}_{k1}^s$ ,  $\hat{\sigma}_{k2}^s$ , and  $\hat{\tau}_k^s (=0)$  in principal directions. Remove the stress from the triangular elements on the equilibrium shape, and compute the unstressed edge lengths  $L_{k1}^0$ ,  $L_{k2}^0$ , and  $L_{k3}^0$ .
- Step 3: Assign a plane P near the target surface, and project the triangular mesh on P to generate the initial mesh of the cutting panel. Let  $L_{k1}^{P}(\mathbf{X})$ ,  $L_{k2}^{P}(\mathbf{X})$ , and  $L_{k3}^{P}(\mathbf{X})$  denote the edge lengths of the triangular elements on P, which are functions of the vector  $\mathbf{X}$  of nodal coordinates of the triangular mesh of P. Solve the following problem to minimize the error in the edge lengths:

Minimize 
$$F(\mathbf{X}) = \sum_{k=1}^{m} \sum_{i=1}^{3} \kappa (L_{ki}^{P}(\mathbf{X}) - L_{ki}^{0})^{2}$$

where  $\kappa$  is a weight parameter. In the following examples,  $\kappa = 1/L_{ki}^0$  to prevent reversal of short edge.

Step 4: Carry out equilibrium shape analysis by minimizing the total strain energy or the total potential energy to find the nodal coordinates on surface and the stresses  $\sigma_{k1}^s$ ,  $\sigma_{k2}^s$ , and  $\tau_k^s$ .

Step 5: Let  $s \leftarrow s+1$ , and modify the reduction stresses  $\hat{\sigma}_{k1}^s$  and  $\hat{\sigma}_{k2}^s$  as

$$\hat{\sigma}_{k1}^{s+1} = \hat{\sigma}_{k1}^{s} + c(\sigma_{k1}^{*} - \sigma_{k1}^{s}), \quad \hat{\sigma}_{k2}^{s+1} = \hat{\sigma}_{k2}^{s} + c(\sigma_{k2}^{*} - \sigma_{k2}^{s}), \quad \hat{\tau}_{k}^{s+1} \ (=0)$$
(11)

Also update the target surface by the equilibrium surface obtained in Step 4.

Step 6: Go to Step 2, if termination condition is not satisfied.

#### 4. Analysis of ETFE film

ETFE film is usually modeled as elasto-plastic material with von Mises yield criterion (Coelho et al. [6], Yoshino and Kato [7]). The relation between stress and strain in uniform tension is often modeled as bilinear relation, which is identified by experiments as shown in Fig. 3, where  $\sigma^{Y}$  and  $\varepsilon^{Y}$  are the yield stress and strain. If the target stress  $\sigma^{*}$  is larger than  $\sigma^{Y}$ , almost uniform stress distribution can be expected, because the stiffness after yielding is smaller than the initial elastic stiffness.



Figure 3: Relation between stress and strain of ETFE sheet under uniform tension.

We consider a monotonic loading process increasing the pressure to reach the equilibrium shape. Although the stiffness after yielding depends on the stress ratio between  $\sigma_{k1}$  and  $\sigma_{k2}$ , we assume the ideal state satisfying  $\sigma_{k1} = \sigma_{k2}$  for which the relation between the equivalent stress and equivalent strain is obtained by experiment. Since we consider a monotonic loading process, the equilibrium shape of an air-pressured ETFE film can be obtained by minimizing  $\Pi(\mathbf{X}) = S(\mathbf{X}) - W(\mathbf{X})$  with

$$S(\mathbf{X}) = \frac{1}{2} \sum_{k=1}^{m} A_k \left[ \boldsymbol{\varepsilon}_k^{YT} \boldsymbol{\sigma}_k^Y + \left( \boldsymbol{\varepsilon}_k - \boldsymbol{\varepsilon}_k^Y \right)^T \left( \boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k^Y \right) \right]$$
(12)

#### 5. Numerical examples

The proposed algorithm of approximate cutting pattern optimization is applied to a frame-supported PVC membrane and an air pressured ETFE film. The optimization problems are solved using sequential quadratic programming implemented in SNOPT Ver. 7 (Gill et al. [8]).

## 5.1. Frame supported PVC membrane

Consider an HP-type frame-supported membrane (Model 1) as shown in Fig. 4. The proportion of the model is  $W_1 = 1.0W$ ,  $W_2 = 1.3W$ , and H = 0.2W. The material property is assumed to be orthotropic elastic. Young's modulus in warp and fill directions are  $2.43 \times 10^2$  kN/m and  $2.27 \times 10^2$  kN/m, respectively. The shear modulus is 24.2 kN/m, and Poisson's ratios are 0.51 and 0.55. The membrane is divided into two cutting sheets as shown in Fig. 5(a). The total numbers of nodes and elements are 160 and 240, respectively.



Figure 4: An HP-type frame supported membrane structure (Model 1).



Figure 5: Cutting sheets of Model 1: (a) triangular mesh, (b) triangular mesh projected to XY-plane before optimization (blue) and cutting sheet after optimization (red).

The target stress is 3.0 kN/m in both warp and weft directions. The history of average, maximum, minimum values and standard deviation of stress is listed in Table 1, where 1 and 2 denote the directions of warp and weft, respectively. As seen from the table, the minimum value increases from a negative value to a positive value. The average value gradually converges to the target value. If we stop at the 20th step, the cutting pattern is as shown in Fig. 5(b). Note that the cutting pattern is close to the triangular plan of the half part of surface, which means that the area of cutting sheet is smaller than the surface area. The stress distribution at the eighth step is shown in Fig. 6. As seen from the figure, the stresses in warp and weft directions are almost uniform except in the area near corners.

	Step 0		Step 5		Step 10		Step 15		Step 20	
Direction	1	2	1	2	1	2	1	2	1	2
Average	0.577	0.102	3.838	3.896	3.479	3.634	3.282	3.371	3.167	3.218
Max.	3.320	2.811	5.341	4.230	5.008	4.011	4.247	3.584	3.803	3.335
Min.	-16.619	-2.680	3.326	3.318	3.170	3.175	3.102	3.104	3.063	3.063
Std. Dev.	2.202	0.621	0.386	0.218	0.314	0.199	0.195	0.113	0.128	0.064

Table 1: History of average, maximum, minimum values and standard deviation



of stress (kN/m) of Model 1.

Figure 6: Stress distribution after 20 steps of optimization: (a) warp direction, (b) weft direction.

#### 5.6. Air pressured ETFE film

Consider a square air-pressured ETFE sheet as shown in Fig. 7, where the ratio of *H* to *W* is 0.058. The elastic material property is isotropic. Young's modulus is  $1.60 \times 10^2$  kN/m, hardening coefficient is 10.4 kN/m, elastic shear modulus is 55.2 kN/m, shear modulus after yielding is 3.60, Poisson's ratio is 0.45, and the stress and strain at yielding is 3.2 kN/m and 0.02, respectively. The specified air pressure is 1.0 kN/m<sup>2</sup>, and the target stress is 4.0 kN/m. In this case, the radius of curvature is  $2 \times 4.0/1.0 = 8.0$  m, if the surface is spherical.



Figure 7: Air-supported ETFE sheet (Model 2).

	Step 0		Step 2		Step 4		Step 7		Step 10	
Direction	1	2	1	2	1	2	1	2	1	2
Average	4.435	4.437	4.108	4.108	4.016	4.017	4.036	4.037	4.072	4.073
Max.	5.100	5.076	4.266	4.283	4.154	4.179	4.503	4.402	4.313	4.449
Min.	2.311	2.360	3.168	3.632	3.721	3.694	3.499	3.316	3.712	3.719
Std. Dev.	0.379	0.362	0.094	0.091	0.085	0.088	0.156	0.160	0.091	0.097

Table 2: History of average, maximum, minimum values and standard deviation of stress of Model 2.

The history of average, maximum, minimum values and standard deviation of stress is listed in Table 2, where 1 and 2 denote the directions in X- and Y-directions on the global coordinates of the cutting sheet. As seen from the table, ETFE has a better accuracy than PVC, because the stiffness at the target stress of ETFE is smaller than that of PVC. The cutting pattern and stress distribution after 10 steps are shown in Figs. 8(a) and (b), respectively. It is seen from these results that the cutting pattern is quite different from the triangular shape, because the curvature of the surface is very large.



Figure 8: Cutting sheets and stress distribution of Model 2: (a) triangular mesh, (b) triangular mesh projected to *XY*-plane before optimization (blue) and cutting sheet after optimization (red).

### 6. Conclusions

An approximate method has been presented for cutting pattern optimization of membrane structures. The conclusions obtained from this study are summarized as follows:

- 1. Approximate plane cutting pattern for the curved surface with specified target stress can be obtained by removing the stress of each triangular element and minimizing the error of edge length for connecting the triangular elements on a plane.
- 2. By adjusting the stress parameter called reduction stress, approximate optimal cutting pattern can be obtained after several iterations of cutting pattern generation and equilibrium shape analysis, which is formulated as an optimization problem of minimizing the strain energy under forced displacements at the boundary.

- 3. The equilibrium shape of a pneumatic membrane structure can also obtained by minimizing the total potential energy including the work done by the air pressure.
- 4. The material property of an ETFE sheet can be modeled as bilinear nonlinear elastic in the process of monotonically increasing the pressure to form the equilibrium shape.

## Acknowledgements

This study is partially supported by JSPS KAKENHI No. 16K14338.

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