Discrete elastica model for shape design of grid shells

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Abstract

A grid shell is designed by bending beams connected by hinge joints. This paper presents an approach to designing grid shell structures made of steel or wood, considering them as an assembly of discretized piecewise linear curves, which are called ‘discrete elastica’. Elastica is defined as the shape of buckled beam-column with large deflection. Therefore, a grid shell consisting of elastica curves can reduce the interaction forces between the curved beams at joints. Shape parameters such as span, height at the support, and height at an internal joint are assigned for the piecewise linear curve. The linear segments are connected with springs at nodes. External moments are also applied to generate various shapes of discrete elastica. We address the problem of designing discrete elastica by minimizing the strain energy function defined with equivalence to the strain energy of a continuous beam. It is shown that the Lagrange multipliers correspond to the reaction forces at supports. In the numerical examples, a gridshell surface with discrete elastica is generated using the proposed method. Large deformation analysis is carried out for verification of the shape generated using discrete elastica.

Keywords

grid shell, discrete elastica, strain energy, large-deformation analysis, active bending

1. Introduction

Grid shell structure (Ohsaki et al. [1], Matsuo et al. [2], Adriaenssens et al. [3]) is one of the most efficient roof structures in view of construction period of time and cost. Shapes of gridshells are generated by straight beams mutually connected by hinge joints. Then forced deformation and external moments are given at the boundary to obtain a curved surface.

A curved surface with uniform grid size is generated by using various existing methods such as compass method and particle-spring method (Tayeb et al. [4], Douthe et al. [5], Bouhaya et al. [6]). However, it is very difficult to generate self-equilibrium shape of a curved surface from plane grid. Furthermore, the interaction forces at joints are too large, if the shape is not appropriate. Hence, the method for reducing the interaction forces between beams have been presented (Ohsaki et al. [1], Matsuo et al. [2]). They defined the target shape of a curved beam as elastic, which is the shape of a buckled beam-column with large deflection. However, to obtain the target shape, we need to calculate a differential equation with respect to the arc-length parameter.

In this study, we present a method for designing shapes of grid shell structures using discrete elastica model (Bruckstein et al. [7], Charamel et al. [8]), which has been studied in the field of computer science. This model enables us to generate easily the target shape of curved beams that are in self-equilibrium state. In addition, we show that a curved surface in three-dimensional space can be generated by connecting several discrete elasticas..

2. Shape design of discrete elastica model

Elastica is defined as a shape of buckled beam-column under point loads at both ends. The deflected equilibrium shape of elastica is obtained by solving a differential equation with respect to the arc-length parameter (Watson and Wang [9]). Since an explicit solution is not possible, the shape of elastica can be obtained using a forward difference approach.

The shape can alternatively be obtained by minimizing an energy function. The bending stiffness of continuous beam, curvatures, and arc-length parameter are denoted by EI, κ, and s, respectively. A parameter β is given to penalize axial deformation. Then, the penalized strain energy function is defined as $\int(EI\kappa^2/2+\beta)\,ds$, which is to be minimized under appropriate boundary and loading conditions. However, it is very difficult to obtain a complex shape composed of multiple curves elastically supported at the ends and connections of curves.

Meanwhile, discrete elastica model is defined as discretized piecewise linear curve with segments that have the same length l as shown in Figure 1. The model has $N+1$ nodes denoted by P_i $(i=0,\ldots,N+1)$. The deflection angle of a segment connecting nodes i and $i+1$ from x-axis, and the angle between the segments $i-1$ and i are denoted by Ψ_i and θ_i $(=\Psi_{i-1}-\Psi_i)$, respectively. By solving an energy minimization problem, we obtain a self-equilibrium shape of discrete elastica model.

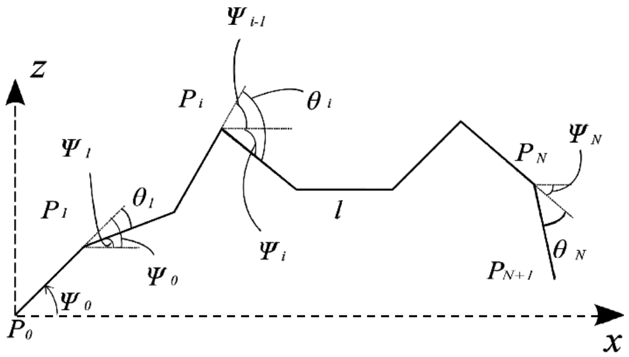


Figure 1: Piecewise linear planar curve with equal-length segments. The total number of segments is N+1 in accordance with Ref. [7].

We assign rotational springs at all nodes and both ends. The springs represent bending stiffness of members and supporting columns. The external moments are given at both ends to generate various shapes. The objective function is the total potential energy consisting of the discretized form of penalized strain energy $\int(EI\kappa^2/2+\beta)\,ds$ and the external work corresponding to the external moments M_0 and M_{N+1}.

The stiffness of each rotational spring is derived from the equivalence of the strain energy between the discrete elastica and the continuous beam. Deformation of the continuous beam, which has stiffness EI and a constant curvature κ, is equivalent to the deformation of discrete elastica model under the condition that the relation between θ and κ is given as $\kappa=\theta/l$ assuming uniform curvature. Hence, the strain energy S is defined as

$$S=\frac{1}{2}EI\kappa^2 l=\frac{1}{2}EI\left(\frac{\theta}{l}\right)^2 l=\frac{EI}{2l}\theta^2\,.$$

Thus, the stiffness of rotational springs is EI/l.

We assign constraints on the span L and the height difference H between the left support P_0 and the right support P_{N+1}. The design variables are $\mathbf{\Psi}\ (= \Psi_0, \dots, \Psi_N)$ and l. The number of nodes and segments are $N+2$ and $N+1$, respectively. The optimization problem, which is an energy minimization problem, is formulated as

$$\min_{\mathbf{\Psi}, l} \quad \Pi\,(\mathbf{\Psi}, \mathrm{l}) = \left[\sum_{i=1}^{N}\left(\frac{EI}{2}\left(\frac{\Psi_i - \Psi_{i-1}}{l}\right)^2 + 1\right)\right] l - M_0\Psi_0 - M_{N+1}\Psi_N \tag{1}$$

subject to

$$\sum_{i=0}^{N} l\cos\Psi_i = L \tag{2}$$

$$\sum_{i=0}^{N} l\sin\Psi_i = H\,. \tag{3}$$

Let λ_1 and λ_2 denote the Lagrange multipliers for constraints (2) and (3), respectively. The Lagrangian is formulated as

$$\mathcal{L}(\mathbf{\Psi}, l, \lambda_1, \lambda_2) = \Pi + \lambda_1\left(\sum_{i=0}^{N} l\cos\Psi_i - L\right) + \lambda_2\left(\sum_{i=0}^{N} l\sin\Psi_i - H\right).$$

Taking derivatives with respect to $N+4$ variables, we have the following stationary conditions.

$$\frac{EI(-\Psi_{i-1} + 2\Psi_i - \Psi_{i+1})}{l^2} - \lambda_1^i\sin\Psi_i + \lambda_2^i\cos\Psi_i = 0, \qquad (i = 1, \dots, N-1) \tag{4}$$

$$\frac{1}{l}\left[\frac{EI(\Psi_1 - \Psi_0)}{l} - M_0\right] - \lambda_1^0\sin\Psi_0 + \lambda_2^0\cos\Psi_0 = 0 \tag{5}$$

$$\frac{1}{l}\left[\frac{EI(\Psi_N - \Psi_{N-1})}{l} - M_{N+1}\right] - \lambda_1^N\sin\Psi_N + \lambda_2^N\cos\Psi_N = 0 \tag{6}$$

$$-\sum_{i=0}^{N} EI\left(\frac{\Psi_i - \Psi_{i-1}}{l}\right)^2 + N + \lambda_1\sum_{i=0}^{N}\cos\Psi_i + \lambda_2\sum_{i=0}^{N}\sin\Psi_i = 0\,. \tag{7}$$

Although the details are omitted, the Lagrange multipliers λ_1 and λ_2 represent the support reaction forces in the horizontal and vertical directions, respectively, and Eqs. (4), (5), and (6) are the equilibrium equations at nodes. The larger β becomes, the shorter the length of the segments becomes. Thus, in the following examples, we set $\beta = 1$ for simplicity.

3. Comparison between discrete and continuous elastica models.

In this section, we compare the shapes obtained by discrete and continuous elastica models. We use sequential quadratic programming available in the library SNOPT Ver.7 (Gill et al. [10]) for energy minimization of the discrete elastica, and we approximate the sensitivity coefficients by finite difference approach.

Abaqus Ver. 6.16 (Dassault Systèmes [11]) is used for large deformation analysis of continuous beams. Forced displacements and external moments are given at both ends of a straight beam on a plane. Length of the beam is equal to the total length of the discrete elastica obtained by optimization. The beam has pin support at the left end, and a forced displacement is given at the right end assuming the translational displacements are constrained by a rigid column. In addition, the external moments, whose values are equal to ones that are applied to the boundaries of discrete elastica model, are given at the both supports.

The material of beam is elastic with Young's modulus 210.0 GPa and Poisson's ratio 0.3. The beam is composed of a plate with width 0.10 m and thickness 0.02 m.

The loading (path) parameter t is increased from 0.0 to 2.0. Upward virtual load equivalent to self-weight is applied to all members from t = 0.0 to 1.0. In the period $1.0 \leq t \leq 2.0$, the virtual load is linearly removed, while the forced displacements and external moments are linearly increased at both ends of the beam. As shown in Fig. 2, continuous elastica model deforms.

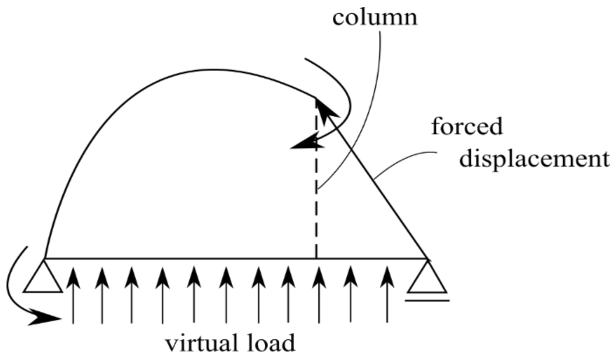
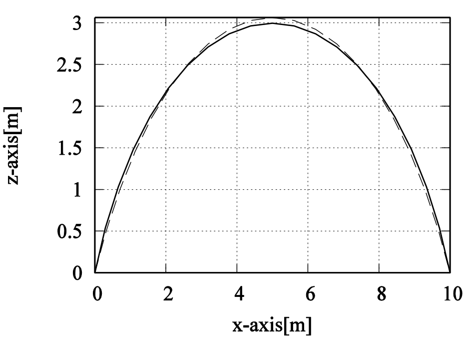


Figure 2: Continuous elastica model

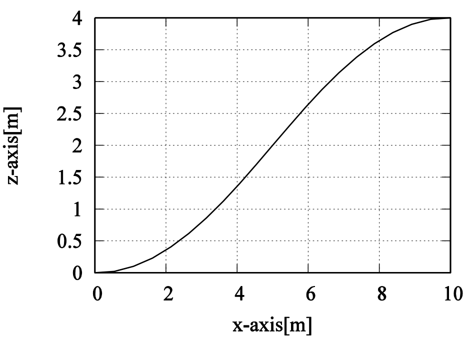
Figures 3(a) and (b) show the results of large deformation analysis with the parameter values in Table 1. The beam is discretized into 20 segments. Shapes of continuous beam and discrete elastica model are expressed, respectively, in solid and dashed lines. The total lengths of models 1 and 2 are 12.281 m and 10.943 m, respectively. It is seen from Figs. 3(a) and (b) that the shapes of beam and discrete elastica model are very close.

Table 1: Parameters of Models 1 and 2

	Model 1	Model 2
M_0 [Nm]	3000	–3000
M_{N+1} [Nm]	–3000	–3000
L [m]	10	10
H [m]	0	4
EI [Nm^2]	14000	14000



(a) Model 1



(b) Model 2

Figure 3: Shapes of Models 1 and 2; dashed line: discrete elastica with 20 segments.

4. Shape design by connecting multiple curves of discrete elastica

We generate shapes of multiple curves of discrete elastica, which are sequentially connected in the same plane. For simplicity, we connect two curves, which has $2N + 3$ nodes denoted by $P_i (= P_0, \dots, P_{2(N+1)})$. The first curve consists of nodes from P_0 to P_{N+1}, and the second curve from P_{N+1} to $P_{2(N+1)}$.

We assign rotational spring at the internal boundary, node P_{N+1}, between the two curves, and define both ends as pin-joint. The rotational spring has very small stiffness α' $(= 0.001)$. We specify span length L for both curves, and heights H_1 and H_2 at the internal boundary and the right support. The optimization problem to find the shape of consisting of two discrete elasticas is formulated as

$$\min_{\boldsymbol{\Psi}, l} \quad \Pi(\boldsymbol{\Psi}, l) = \left[\sum_{i=1}^{2N+1} \left(\frac{EI}{2} \left(\frac{\Psi_i - \Psi_{i-1}}{l} \right)^2 + 1 \right) - \left(\frac{EI}{2} - \alpha' \right) \left(\frac{\Psi_{N+1} - \Psi_N}{l} \right)^2 \right] l - M_0 \Psi_0 - M_{2(N+1)} \Psi_{2N+1} \tag{8}$$

subject to

$$\sum_{i=0}^{N} l \cos \Psi_i = L \tag{9}$$

$$\sum_{i=0}^{N} l \sin \Psi_i = H_1 \tag{10}$$

$$\sum_{i=N+1}^{2N+1} l \cos \Psi_i = L \tag{11}$$

$$\sum_{i=N+1}^{2N+1} l \sin \Psi_i = -H_1 + H_2 \,. \tag{12}$$

5. Comparison between connected discrete and continuous elasticas

We solve the optimization problem with parameter values in Table 2 in order to design shapes of connected models, and compare the obtained shapes with continuous beams. The total lengths of models 3 and 4 are 21.615 m and 22.074 m, respectively.

The left end of the connected discrete elastica is pin supported, the right end is roller supported, and the internal joint is fixed in the horizontal direction. For large deformation analysis, the stiffness of the internal joint of two beams is equal to that of the discrete elastica models. The deformation of connected continuous elastica model is shown in Fig. 4. Figures 5(a) and (b) show the results of large deformation analysis. It is seen from these figures that the shapes of discrete and continuous elastica modes are very close.

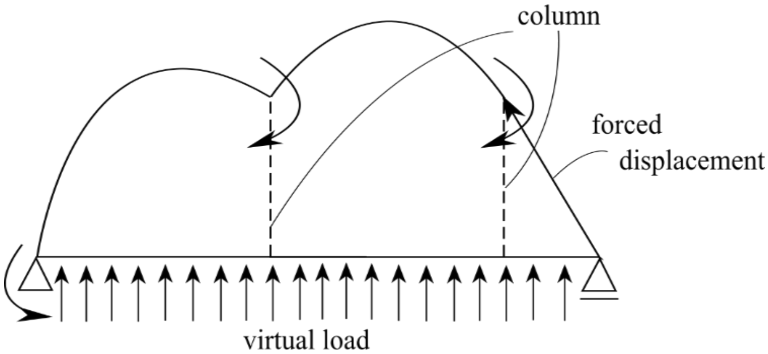
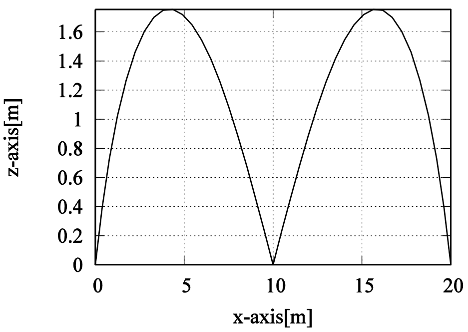


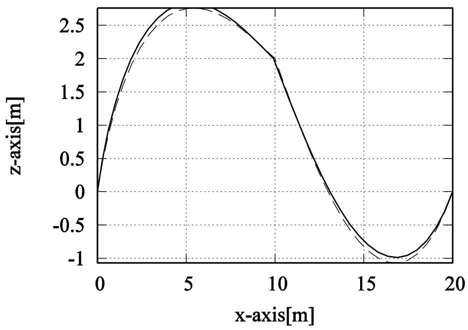
Figure 4: Connected continuous elastica model

Table 2: Parameters of models 3 and 4

	Model 3	Model 4
M_0 [Nm]	3000	3000
$M_{2(N+1)}$ [Nm]	-3000	3000
$2L$ [m]	20	20
H_1 [m]	0	2
H_2 [m]	0	0
EI[Nm^2]	14000	14000



(a) Model 3



(b) Model 4

Figure 5: Shapes of Models 3 and 4; dashed line: discrete elastica with 40 segments.

6. Generating surfaces with discrete elastica model

In this section, we generate a target surface of gridshell using discrete elastica. Furthermore, we compare the target shape with the shape of grid shell, which is generated by forced displacements and external moments to the continuous beams connected with lateral beams.

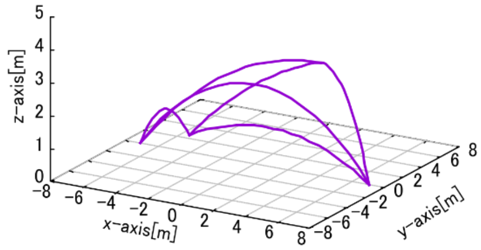
Figure 6(a) shows the target surface generated by connecting six elasticas. The plan of surface is a 10 m $\times$10 m square. The parameter values are listed in Table 3. Model (a) represents the curves along the exterior boundary, whose height difference is 2 m. Models (b) and (c) represent the curves along the diagonal lines, which intersect at the center. The both ends of (b) are on the ground and the both ends of (c) are connected to columns with the height 2 m.

Figure 6(b) shows the target shape of gridshell, which is generated by assigning boundary conditions and external moments to the primary beams in Fig. 6(a). The primary beams have the width 0.120 m and thickness 0.020 m. Meanwhile, we assign slender secondary beams, which have the width 0.040 m and thickness 0.015 m, to reduce the interaction forces at nodes.

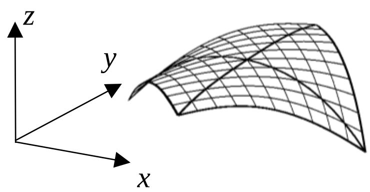
Figures 6(c)-(f) show that the shapes of beams and discrete elasticas are very close. As the result of large deformation analysis, all member stresses are confirmed to be smaller than the yield stress 325 MPa. Table 4 shows reaction forces at supports.

Table 3: Parameters of model (a), (b) and (c)

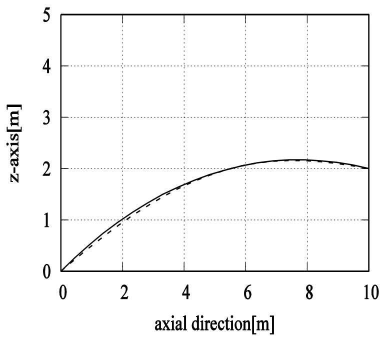
	Model (a)	Model (b)	Model (c)
M_0 [Nm]	1100.0	1335.5	286.0
M_{N+1} [Nm]	-1100.0	-1335.5	-286.0
L [m]	10	$10\sqrt{2}$	$10\sqrt{2}$
H [m]	2	0	0
EI [Nm^2]	16800	16800	16800



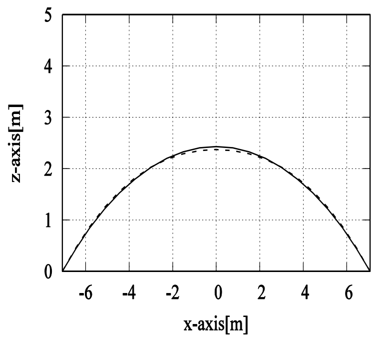
(a) The target surface



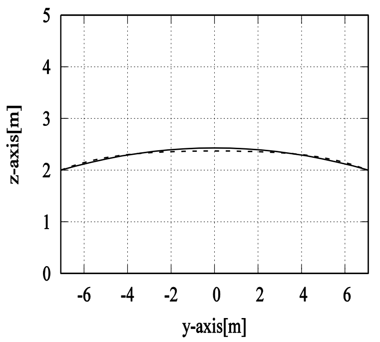
(b) Grid shell composed of continuous beams



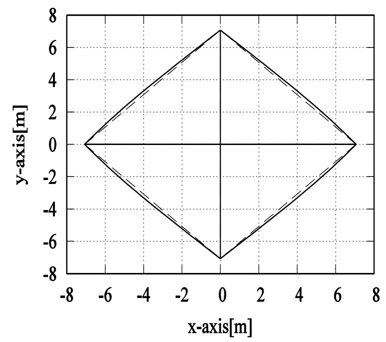
(c) Model (a)



(d) Model (b)



(e) Model (c)



(f) Plan

Figure 6: The comparison between the target shape and grid shells composed of continuous beams. These beams have 20 segments. (f) shows the plans of discrete and continuous elasticas. Continuous elastica is three dimensional shape. Thus, it is deflected in *x*-*y* direction.

Table 4: Reaction forces

	Model (a)	Model (b)	Model (c)
λ_1 [N]	34.7	62.1	1.37
λ_2 [N]	6.9	0.0	0.0

7. Conclusions

1. Curved beams of gridshell can be modeled as discrete elastica, which is composed by rigid bars and rotational springs. The stiffness of each rotational spring is derived from the equivalence of the strain energy between the discrete and continuous elastica models.

2. The shape of discrete elastica can be found by solving an optimization problem. The objective function is defined as the total potential energy consisting of discretized form of penalized strain energy and the external work corresponding to the external moments. Solving the problem under the constraints with the span length and the height of supports, we can generate various shapes of discrete elastica. Furthermore, the reaction forces needed for deformation can be found from the Lagrange multipliers.

3. Target shape of a curved surface of gridshell can be obtained by connecting several discrete elasticas with thin secondary beams. It has been confirmed by large deformation analysis that the surface shape obtained by assigning forced displacements and external moments is close to the target shape defined using the discrete elasticas. Therefore, we can expect that the interaction forces at the connections are very small.

Acknowledgement

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