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An optimization method for generating self-equilibrium shape of curved surface from developable surface

Jinglan CUI *, Makoto OHSAKI[a]

*Department of Architecture and Architectural Engineering, Kyoto University,
Kyoto-Daigaku Katsura, Nishikyo, Kyoto 615-8540, Japan, cui.jinglan.63w@st.kyoto-u.ac.jp
[a] Department of Architecture and Architectural Engineering, Kyoto University

Abstract

Membrane structure is generated by stretching and bending a plane membrane sheet. Therefore, the self-equilibrium shape of the membrane structure may be close to a developable surface, although the self-equilibrium shape with uniform tension has a negative Gaussian curvature. In this study, we propose a method to generate a membrane structure from a developable surface. The method consists of generation of developable surface, projection of the surface to a plane, correction of cutting pattern by adjustment of edge lengths of triangular elements, reduction of target uniform stress, and generation of an equilibrium shape through minimization of strain energy. Effectiveness of the proposed method is demonstrated through examples of a simple model and a complex model.

Keywords: developable surface, membrane structure, cutting pattern, optimization.

1. Introduction

Recently, a number of designers have focused on free-form surface to generate a free architectural form which is different from the analytical curved surfaces such as cylindrical surfaces and spherical surfaces. In order to optimize the shape of free-form shell, a method of connecting developable surfaces was proposed (Nakamura *et al.* [1]). In this study, we extend it to shape design (Barnes [2]) of membrane structures.

Developable surface is a kind of special shape of the ruled surface formed by moving a generating line along a directing curve. It can be generated by assigning a developability condition to the ruled surface so that the direction of normal vector of surface does not change along the generating line. In addition, since one of the principal curvatures of a developable surface is zero, it can also be generated by giving a condition that the Gaussian curvature vanishes on the whole curved surface.

Hyperbolic paraboloid is one of ruled surface and is often used as a prototype curved surface of a membrane structure. The ruled surface generally has positive and negative principal curvatures except developable surface; therefore, Gaussian curvature is negative. Such curved surface is obtained by giving the torsional deformation to developable surface. Since membrane structure is generated by connecting plane cutting patterns, its equilibrium shape is close to a developable surface (Meek and Tan [3]).

In this study, we use the developable surface composed by Bézier curve as the initial shape, and propose a method for finding a cutting pattern from a developable surface. This method will be realized through the following operations. First, we generate a reasonable developable surface structural form of a shell through optimization for the purpose of uniformly distributed stress (Duysinx and Bendsøe [4]). In order to convert the developable surface to an equilibrium shape of membrane structure, the cutting pattern is optimized so that the edge lengths of triangular elements on the plane cutting sheet is close to those on the target developable surface (Ohsaki [5], Ohsaki *et al.* [6]). Next, by removing the strain due to ideal uniform membrane stress, we obtain the cutting pattern so that a uniform stress is generated in the membrane structure. Based on the final cutting pattern, the equilibrium shape of membrane structure (Harber and Abel [7]) is obtained by carrying out forced-displacement analysis through minimization of the strain energy (Bouzidi and Le van [8], Mosler [9]).

2. Generation of developable surface

2.1. Representation of the developable surface by Bézier curve

It is assumed that the target curved surface, which is defined by the developable surface (Nakamura *et al.* [1]), and the planar cutting pattern are both triangulated with the same topology. The shape of the cutting pattern can be obtained by minimizing the weighted sum of the squares of differences between the edge lengths of triangles of the planar cutting pattern and the curved surface.

The variables of the optimization problem are the (x, y) coordinates of the nodes on the cutting pattern. Three components of corner nodes are fixed to prevent rigid body translation and rotation. Moreover, in order to prevent reversal of normal direction of triangle and existence of crossing edges, the upper and lower bounds are given to the range of design variables. Let l_{i1}^0, l_{i2}^0 and l_{i3}^0 denote the edge lengths of the *i*-th triangle on the developable surface. The corresponding edge lengths on the cutting pattern are denoted by l_{i1}, l_{i2}, and l_{i3}. When the developable surface is composed of *m* nodes and *n* elements, the shape of the cutting pattern can be found by solving the following optimization problem (Ohsaki and Fujiwara [10]).

$$\begin{cases} \min\limits_{Q} \ F(\mathbf{Q}) = \sum_{i=1}^{n} \sum_{j=1}^{3} \omega_{ij} \cdot (l_{ij} - l_{ij}^0)^2 \\ s.t. \quad x_k^L \le x_k \le x_k^U \\ \qquad\ \ y_k^L \le y_k \le y_k^U \qquad (k = 1, 2, \cdots m) \end{cases} \tag{1}$$

where $\mathbf{Q}$ is the vector of nodal coordinates (x_k, y_k), which are the design variables, ω_{ij} is the weight coefficient, (x_k^L, y_k^L) and (x_k^U, y_k^U) are the lower and upper bounds of (x_k, y_k), respectively. SNOPT Ver. 7 (Gill *et al.* [11]) is used for solving optimization problems.

The membrane material is modeled as an orthotropic material, and the coordinates of the principal directions are assumed to coincide with (x, y), for simplicity. Define E_x and E_y as the elastic modulus in x and y directions, G as the shear modulus, ν_{xy}, ν_{yx} as Poisson's ratio, and suppose $E_x \ge E_y$. By using $\gamma = E_x / E_y$ and $k = G / E_y$, the element stress $\boldsymbol{\sigma} = \left\{\sigma_x^e, \sigma_y^e, \tau_{xy}^e\right\}^T$ and the element strain $\boldsymbol{\varepsilon} = \left\{\varepsilon_x^e, \varepsilon_y^e, \gamma_{xy}^e\right\}^T$ have as the following relation:

$$\boldsymbol{\sigma} = \boldsymbol{D} \cdot \boldsymbol{\varepsilon} \tag{2}$$

$$\boldsymbol{D} = \frac{E_y}{1 - \gamma \nu_{yx}^2} \begin{bmatrix} \gamma & \gamma \nu_{yx} & 0 \\ \gamma \nu_{yx} & 1 & 0 \\ 0 & 0 & k(1 - \gamma \nu_{yx}^2) \end{bmatrix} \tag{3}$$

The uniform strain $\boldsymbol{\varepsilon}_0 = \left\{\varepsilon_x^e, \varepsilon_y^e, \gamma_{xy}^e\right\}^T$ of the element is determined by introducing an ideal uniform stress $\boldsymbol{\sigma}_0 = \left\{\sigma_0^e, \sigma_0^e, 0\right\}^T$ in equation (2) into the cutting pattern. As a result, the reduction ratio of the cutting pattern coordinates is obtained, and the cutting pattern is updated by $x_p = x(1 - \varepsilon_x)$ and

$y_p = y(1-\varepsilon_y)$, where (x, y) is the coordinates of all nodes on the cutting pattern before removing the target stress, and (x_p, y_p) is the coordinates of all nodes in the unstressed state.

2.2. Generation of membrane structure

When the developable surface is simply regarded as a membrane structure, it is not an equilibrium shape of the membrane structure obtained by connecting the membrane sheet to the boundary of the surface. We find the equilibrium shape of membrane structure by minimizing the strain energy of the membrane elements due to the forced deformation at along the boundary. The optimization problem is formulated as the strain energy minimization problem as

$$\begin{cases} \min\limits_{Q} \ E(\mathbf{Q}) = \dfrac{1}{2}\int_v \left(\boldsymbol{\varepsilon}^T \boldsymbol{D}\boldsymbol{\varepsilon}\right) dV \\ s.t. \quad \mathrm{X}_k^L \le \mathrm{X}_k \le \mathrm{X}_k^U \\ \qquad \mathrm{Y}_k^L \le \mathrm{Y}_k \le \mathrm{Y}_k^U \\ \qquad \mathrm{Z}_k^L \le \mathrm{Z}_k \le \mathrm{Z}_k^U \qquad (k = 1, 2, \cdots m) \end{cases} \tag{4}$$

where $\mathbf{Q}$ represents the vector of nodal coordinates $(\mathrm{X}_k, \mathrm{Y}_k, \mathrm{Z}_k)$ $(k = 1, 2, \cdots m)$ on equilibrium surface of membrane structure, $(\mathrm{X}_k^L, \mathrm{Y}_k^L, \mathrm{Z}_k^L)$ and $(\mathrm{X}_k^U, \mathrm{Y}_k^U, \mathrm{Z}_k^U)$ are the lower and the upper bounds for $(\mathrm{X}_k, \mathrm{Y}_k, \mathrm{Z}_k)$.

3. Generation method of membrane structure from developable surface

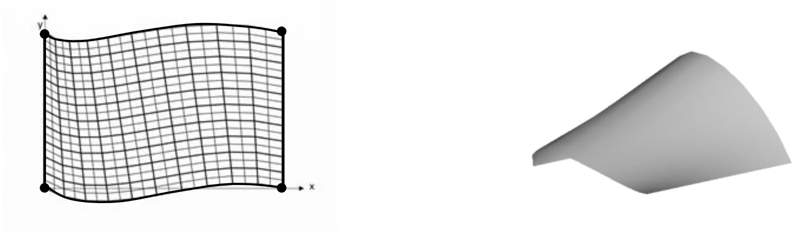
After designing the developable surface in view of architectural form and mechanical property, the membrane structure close to the developable surface can be realized by the following operations:

1) The boundary of the initial membrane structure is expressed by Bézier curve according to the architectural design and structural conditions.
2) Replacing the membrane material to a general stiff material, the control points of Bézier curves are optimized under constraints on developability conditions and mechanical properties.
3) The initial cutting pattern is created by projecting the obtained developable surface on a plane.
4) The nodal coordinates of the cutting sheets of membrane are obtained by solving the optimization problem (1).
5) Introducing constant stress into the triangular elements of the cutting pattern, the unstressed shape of cutting pattern is obtained by removing the stress.
6) The equilibrium shape of membrane structure is obtained by forced deformation analysis, which is formulated as a minimizing problem (4) the total strain energy.

4. Examples

4.1. Simple model

In order to determine the membrane structure form, the target shape of developable surface is generated using the method in Nakamura *et al.* [1] as shown in Figs. 3(a), (b). For a shell with a rectangular plan of $60.0 \times 40.0\mathrm{m}$, we fix the positions of the four corners, and model the developable surface with two cubic Bézier curves as directing curves.



(a) plan view (b) diagonal view

Figure 3 : Optimal developable surface

The developable surface is first projected onto plane as shown in Fig. 4(a). The material constants of the membrane are listed in Table 1. The planar cutting pattern obtained by solving problem (1) is shown in Fig. 4(b). Introducing an ideal target tensile stress of 3.0 kN/m into this cutting pattern and shrinking it, the unstressed cutting pattern is obtained as shown with solid lines in Fig. 4(c). The cutting pattern in Fig. 4(b) is also plotted with dotted lines in Fig. 4(c), which shows that the unstressed cutting pattern is slightly smaller than that of the stressed one in Fig. 4(b).

Table 1 Material constants of membrane

Elastic coefficient	E_x=243KN/m	E_y=227KN/m
Poisson's ratio	V_{xy}=0.55	V_{yx}=0.51
Shear rigidity	G_{xy}=76.84KN/m	G_{yx}=76.84KN/m

(a) projected developable surface (b) cutting pattern after length optimization

(c) unstressed cutting pattern (d) optimal membrane structure

Figure. 4 Cutting pattern generation and equilibrium surface of membrane

After obtaining the cutting pattern, the equilibrium membrane structure is generated by solving problem (4) as shown in Fig. 4(d). As can be seen from Table 2, the average membrane stress is a little larger than the target stress, although the minimum values are negative, which indicates that slackening occurs. Since the equilibrium shape of membrane structure and the developable surface are different, we cannot expect a better result without slight modification of the shape of the cutting pattern. Using multiple membrane sheets will also lead to a better result.

Table 2 Minimum, maximum, and average stress (kN/m) of the equilibrium surface

	Minimum	Maximum	Average
σ_x	1.012	9.621	4.387
σ_y	0.127	8.724	2.892

The obtained membrane structure has a slight difference from the shape of the developable surface as shown in Fig. 5. The minimum value of the difference of nodal coordinates at *x*, *y* and *z* direction are –0.270, –0.373 and –0.068, the maximum value are 0.069, 0.025 and 1.040, the average value are –0.057, –0.011 and 0.336 respectively. The maximum difference of the curve at the center from the straight line (*z* direction) of generatrix is 1.04 m, which is about 2.6 % of the span length.

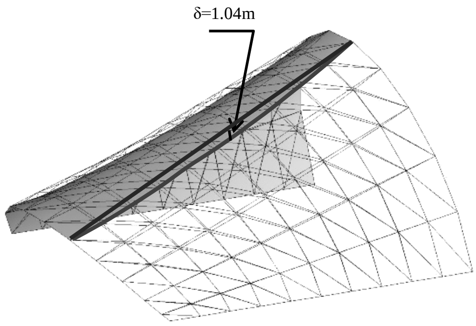
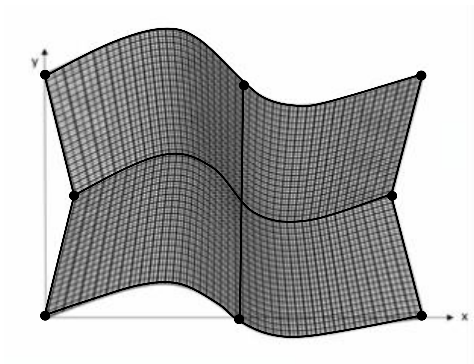


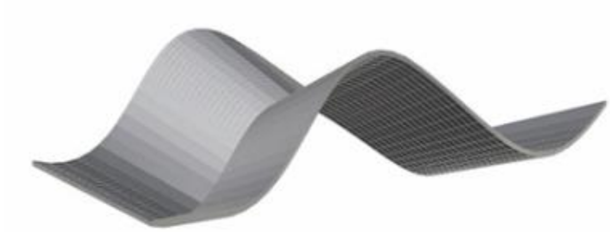
Figure. 5 Comparison of developable surface and membrane structure

4.2. Complex model

Consider a developable surface obtained by optimization on a 60.0×40.0 m plane area using three Bézier curves. Figure 6(a) is the plan of the developable surface, and Fig. 6(b) is the diagonal view of the surface, where the locations of eight supports are fixed. When generating a membrane structure, stiff frameworks are assumed to exist, as shown with thick solid lines in Fig. 6(a), along both sides of the three directing curves and six generating lines including the internal boundaries.



(a) plan view

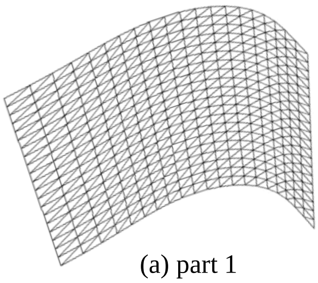


(b) diagonal view

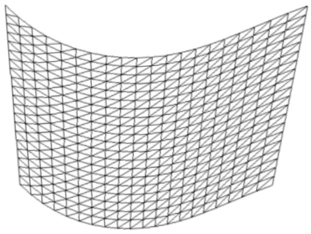
Figure 6 : Optimal developable surface

We partition the developable surface into four parts and generate the cutting pattern by solving problem (1). The material constants of the membrane and the target stress are the same as those of the simple model. Figure 7 shows the cutting patterns of the four parts. By minimizing the strain energy, a membrane structure is obtained as shown in Fig. 8.

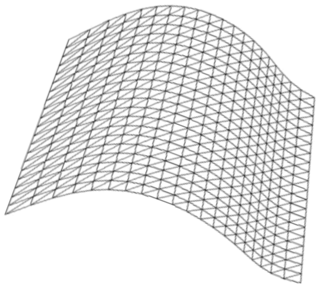
As seen from Table 3, the average membrane stresses in x and y directions are close to the target stress 3.0 kN/m.



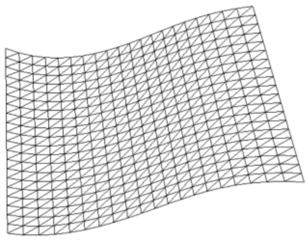
(a) part 1



(b) part 2



(c) part 3



(d) part 4

Figure7: Cutting pattern of four parts

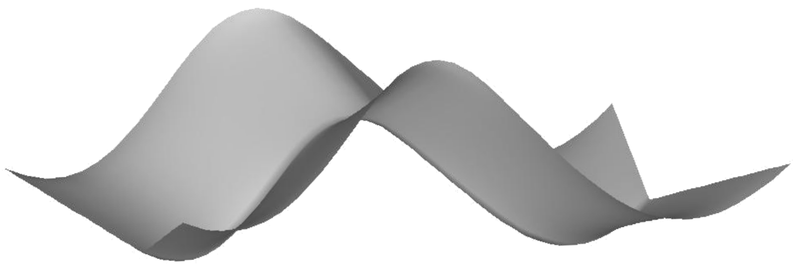


Figure8 Membrane structure generated from developable surface

Table 3 Minimum, maximum, average (kN/m) stress of the equilibrium surface

	Minimum	Maximum	Average
σ_x	1.106	8.265	3.206
σ_y	0.599	7.400	2.589

The minimum values of the difference of nodal coordinates between the developable surface and the membrane surface in *x*, *y* and *z* directions are –0.270, –0.203 and –0.571, the maximum values are 0.218, 0.159 and 0.294, the average values are –0.003, –0.020 and –0.140, respectively. As seen in Fig.9, the maximum difference from curve center to the straight line of generatrix is 0.294 m, which is approximately 0.7% of the span length.

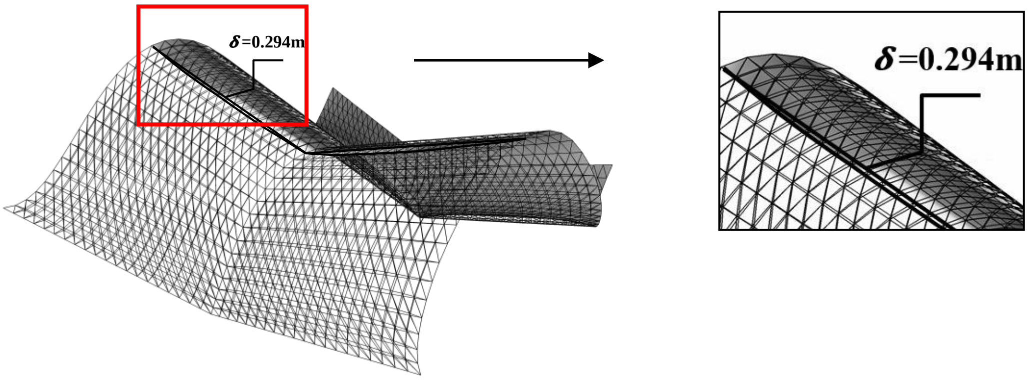


Figure. 9 Comparison of developable surface and membrane structure

5. Conclusion

In this study, the method for finding a developable surface is extended to generate cutting pattern and equilibrium shape of membrane structure.

First, a developable surface is generated using the method by Nakamura *et al.* [1]. Then the developable surface is triangulated and it is expanded to a plane by minimizing difference of the edge lengths of the triangulated elements. The cutting pattern corresponding to the ideal target stress is obtained by removing the stress. Finally, shape of the equilibrium surface of membrane is generated by minimizing the strain energy.

Since the in-plane deformation of the membrane surface is very small, the average values of stress of the membrane equilibrium surface are close to the target values. However, there is a negative value in the minimum stress; therefore, it is necessary to slightly modify the shape of the cutting pattern using the method by Ohsaki *et al.* [6]. The stress distribution will also be improved if multiple membrane sheets are used. Furthermore, because the Gaussian curvature of the membrane surface with uniform tension is 0, the shape obtained by this method is slightly different from the initial target developable shape.

Acknowledgement

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