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Shape optimization of free-form shells consisting of developable surfaces

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Abstract

A method is presented for optimization of free-form shells consisting of developable surfaces. The surface is modelled as 2×1 or 3×1 tensor product Bézier surfaces. The developability conditions are assigned numerically as constraints for the optimization problem for minimizing the strain energy under self-weight as the mechanical performance. The control points of the Bézier curves are considered as design variables. A new continuity condition of normal vector of surface is proposed for connecting ruled surfaces. Optimal solutions are found using a nonlinear programming approach, and a finite difference approximation is used for sensitivity analysis. The effectiveness of the present approach is confirmed through various examples and the characteristics of the optimal shapes are discussed.

Keywords: developable surface, free-form shell, shape optimization, Bézier surface

1. Introduction

Recently, free-form shells that are not categorized into traditional analytical surfaces are extensively designed and constructed owing to the advancement of computer technologies as well as development of structural materials and construction methods (Adriaessens et. Al [1]). However, free-form shells are very costly, and demand very long construction period.

To reduce the cost for construction of free-form shells, the second author developed a design method using ruled surface, which is generated by translation and rotation of a line, and has zero Gaussian curvature everywhere on the surface (Seki et. Al [2], Pottmann and Wallner [3]). Using a ruled surface, the shell surface is defined as an assemblage of lines, and the cost for mold for a reinforced concrete shell may be reduced. However, the generating lines of a ruled surface has torsion; therefore, the surface cannot be constructed from a plate without in-plane deformation.

In this study, we present an optimization method for designing free-form shells consisting of developable surfaces. The surface is modeled as 2×1 or 3×1 tensor product Bézier surfaces. The developability conditions are assigned numerically as constraints for the optimization problem for minimizing the strain energy under self-weight as the mechanical performance. The control points of the Bézier curves are considered as design variables. A new continuity condition of normal vector of surface is proposed for connecting developable surfaces. To avoid an erroneous solution and divergence in the analysis and optimization processes, a constraint is given for material volume in terms of the area of the surface. We use nonlinear programming approach, and a finite difference approximation is used for sensitivity analysis. The effectiveness of the present approach is confirmed through various examples of concrete shells, and the characteristics of the optimal shapes are discussed.

2. Definition of developable surface

A developable surface is generated by bending a planar sheet without stretching, warping, or tearing. It is a special class of ruled surface that has a generating line moving along a directing curve. Therefore, if we use a developable surface for designing a free-form shell, the surface can easily be generated from a plane, although variation of the shape is restricted. Furthermore, generating line of a developable surface does not have torsion, which is contrary to a general rule surface; therefore, scaffolding of a concrete shell becomes even easier.

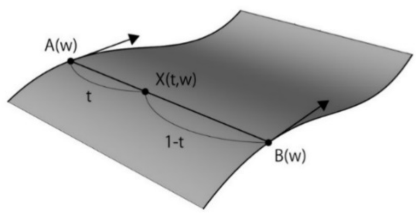


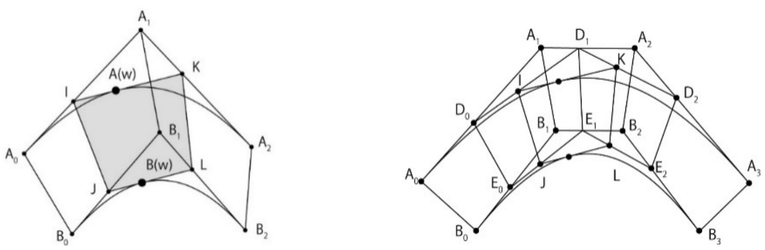
Figure 1: Example of a ruled surface.

An example of ruled surface is shown in Fig. 1. A developable surface is defined by adding the developability condition to a ruled surface. Let $\mathbf{A}(w)$ denote a directing curve in 3-dimensional space with parameter w. Another curve with parameter w is denoted by $\mathbf{B}(w)$. Then a ruled surface $\mathbf{X}(t,w)$ is defined as

$$\mathbf{X}(t,w) = (1-t)\mathbf{A}(w) + t\mathbf{B}(w), \quad (0 \le t \le 1) \tag{1}$$

where t is the parameter for a generating line that connects the two points on the curves with the same value of w.

We use a Bézier curve for modeling the directing curve. Since a ruled surface has a generating line between directing curves, the surface can be modeled using a so called $(n,1)$- Bézier patch, which has order 1, i.e., a line, in one direction.



(a) (b)

Figure 2: Developable surfaces using Bézier curves; (a) order 2, (b) order 3.

The developability condition is defined such that the tangent lines of curves $\mathbf{A}(w)$ and $\mathbf{B}(w)$ at the same parameter value exists on the same plane, which implies that the tangent plane remains the same along the generating line. This condition is formulated as (Chu and Sequin [4])

$$\dot{\mathbf{A}}(w)\times\dot{\mathbf{B}}(w)\cdot[\mathbf{A}(w)-\mathbf{B}(w)]=0 \tag{2}$$

where a dot denotes differentiation with respect to w. If Eq. (2) is satisfied for any w between 0 and 1, then the surface is developable.

By defining the directing curve by a Bézier curve, various shapes can be generated using small number of variables. Developable surfaces using Bézier curves of orders 2 and 3 are shown in Fig. 2 (a) and (b), respectively, together with their control points. The developability conditions are written using $\vec{IJ}$, $\vec{KL}$, and $\vec{IK}$ as

$$\vec{IJ}\cdot\vec{KL}\times\vec{IK}=0 \tag{3}$$

For Bézier curve of order n, the developability condition (3) is expressed by polynomials of order $3(n-1)$ with respect w.

Let $\mathbf{a}_i=\mathbf{A}_i-\mathbf{A}_{i-1}$ and $\mathbf{c}_i=\mathbf{B}_i-\mathbf{A}_i$ $(i=0,1,\ldots)$. Using the Cox de Boor recurrence relation, the developability condition for Bézier curves of orders 2 and 3 are written as (Chu and Sequin [4])

- Order 2:

$$\begin{aligned}
&\mathbf{a}_1\cdot\mathbf{c}_0\times\mathbf{c}_1=0\\
&\mathbf{a}_2\cdot\mathbf{c}_1\times\mathbf{c}_2=0\\
&\mathbf{a}_1\cdot\mathbf{c}_0\times\mathbf{c}_2+\mathbf{a}_2\cdot\mathbf{c}_0\times\mathbf{c}_1=0\\
&\mathbf{a}_1\cdot\mathbf{c}_1\times\mathbf{c}_2+\mathbf{a}_2\cdot\mathbf{c}_0\times\mathbf{c}_2=0
\end{aligned} \tag{4}$$

- Order 3:

$$\begin{aligned}
&\mathbf{a}_1\cdot\mathbf{c}_0\times\mathbf{c}_3+\mathbf{a}_2\cdot\mathbf{c}_0\times\mathbf{c}_1=0\\
&\mathbf{a}_2\cdot\mathbf{c}_2\times\mathbf{c}_3+\mathbf{a}_3\cdot\mathbf{c}_1\times\mathbf{c}_3=0\\
&\mathbf{a}_1\cdot\mathbf{c}_0\times\mathbf{c}_3+3\mathbf{a}_1\cdot\mathbf{c}_1\times\mathbf{c}_2+4\mathbf{a}_2\cdot\mathbf{c}_0\times\mathbf{c}_2+\mathbf{a}_3\cdot\mathbf{c}_0\times\mathbf{c}_1=0\\
&\mathbf{a}_1\cdot\mathbf{c}_2\times\mathbf{c}_3+4\mathbf{a}_1\cdot\mathbf{c}_1\times\mathbf{c}_3+\mathbf{a}_3\cdot\mathbf{c}_0\times\mathbf{c}_3+3\mathbf{a}_3\cdot\mathbf{c}_1\times\mathbf{c}_2=0\\
&\mathbf{a}_1\cdot\mathbf{c}_1\times\mathbf{c}_3+\mathbf{a}_2\cdot\mathbf{c}_0\times\mathbf{c}_3+3\mathbf{a}_2\cdot\mathbf{c}_1\times\mathbf{c}_2+\mathbf{a}_3\cdot\mathbf{c}_0\times\mathbf{c}_2=0
\end{aligned} \tag{5}$$

3. Optimization problem

Consider a shell roof subjected to the self-weight and live load. The shell is discretized into quadrilateral finite elements. Let $\mathbf{K}(\mathbf{x})$ denote the stiffness matrix, which is a function of the design variable vector $\mathbf{x}$ consisting of the coordinates of control points of Bézier curves.

The displacement vector $\mathbf{u}(\mathbf{x})$ against the load vector $\mathbf{p}(\mathbf{x})$ is found by solving the stiffness equation

$$\mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x})=\mathbf{p}(\mathbf{x}) \tag{6}$$

The total strain energy $f(\mathbf{x})$ is defined as

$$f(\mathbf{x})=\frac{1}{2}\mathbf{u}^T(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{7}$$

The material of shell is concrete with Young's modulus 24 GPa, Poisson's ratio 0.2, and weight density 24.5 kN/m^3. The thickness of shell is 0.2 m. We use OpenSees (PEERC [5]) for analysis, and the shell element is MITC4. A library SNOPT ver. 7.2 (Gil et. al [6]) utilizing sequential quadratic programming is used for optimization. The sensitivity coefficients are approximated by central difference approach.

As noted in Sec. 2, the developability condition (3) of a surface defined by Bézier curve of order n is expressed by a polynomial of order $3(n-1)$ with respect w. Therefore, the condition is satisfied for any w between 0 and 1, if Eq. (3) is satisfied at $3(n-1)+1=3n-2$ different values of w. This way, the complex analytical forms in Eqs. (4) and (5) are avoided, and optimal developable surfaces can be found numerically.

The following constraint is given for the surface area $S(\mathbf{x})$ to prevent a stiff shape owing to small area of shell:

$$S(\mathbf{x}) \geq \mathrm{S}_0 \tag{8}$$

where S_0 is the area of the initial solution.

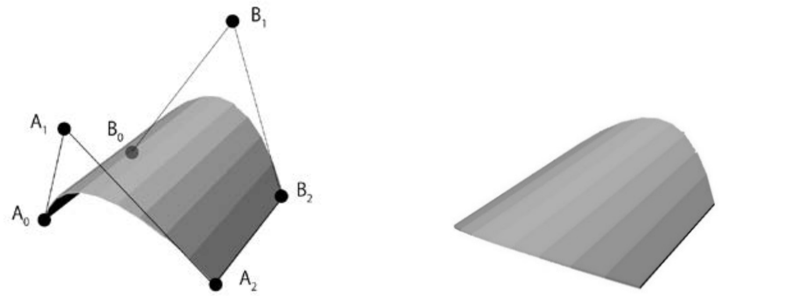
Since the number of control points of Bézier curve of order n is $n+1$, the bound of number variables of a developable surface is $3(n+1)\times 2-6-(3n-2)=3n+2$.

4. Optimization results of developable surfaces using Bézier curve of orders 2 and 3

To confirm applicability of the proposed numerical assignment of developability conditions, we obtain optimal shapes using the following two types of developability conditions for Bézier curve of order 2:

1. Four conditions (4) are to be satisfied.
2. The condition (3) is satisfied at four points $w=0.2$, 0.4, 0.6, and 0.8.

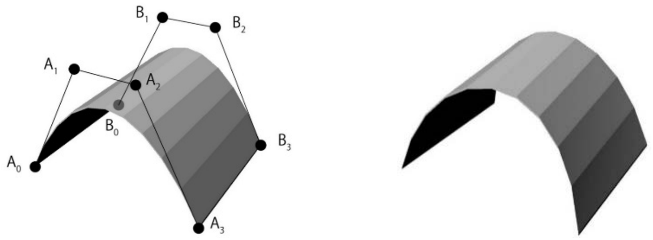
The initial shape is shown in Fig. 3(a) with the control points. The span lengths in the directions of generating line and directing curve are 50 m and 60 m, respectively, and the locations of points A_0, A_2, B_0, and B_2 are fixed. The three coordinates of points A_1 and B_1 are the design variables. It has been confirmed that the same solution as shown in Fig. 3(b) is obtained using the above two approaches.



(a) initial shape; $f(\mathbf{x})=731$ kNm. (b) optimal shape; $f(\mathbf{x})=514$ kNm.

Figure 3: Optimal developable surface using Bézier curve of order 2.

The initial and optimal shapes of the model using Bézier curves of order 3 are shown in Figs. 4(a) and (b), respectively. It is seen form the figure that a round shape is generated through optimization, and the strain energy is reduced drastically to 16% of the initial shape.

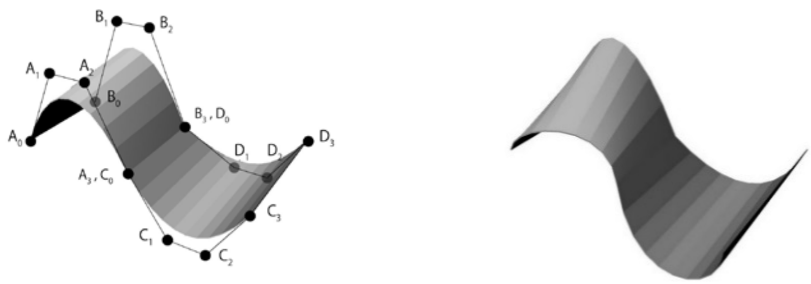


(a) initial shape; $f(\mathbf{x}) = 1025$ kNm (b) optimal shape; $f(\mathbf{x}) = 167$ kNm

Figure 4: Optimal developable surface using Bézier curve of order 3.

5. Optimization results of developable surfaces using assemblage of developable surfaces

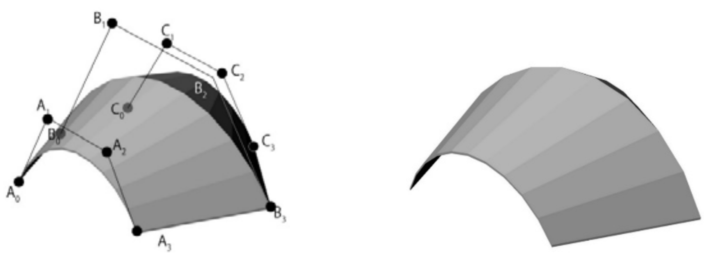
Consider first the case, where two surfaces are connected in the direction of directing curves, as shown in Fig. 5(a), defined by Bézier curves of order 3. The span lengths in the directions of directing curve and generating line are 120 m and 50 m, respectively, except the distance between points C3 and D3, which is 70 m. We may simply assign continuity of tangent vectors of Bézier curves. However, we release the condition so that the normal vector is continuous. This condition is achieved by assigning the quadrilaterals (A_2, A_3, D_0, D_1), (B_2, B_3, C_0, C_1), and (A_2, B_2, C_1, D_1) are in the same plane. Using this condition, the optimal shape is obtained as shown in Fig. 5(b).



(a) initial shape; $f(\mathbf{x}) = 2094$ kNm (b) optimal shape; $f(\mathbf{x}) = 748$ kNm

Figure 5: Optimal developable surface connected in direction of direction lines.

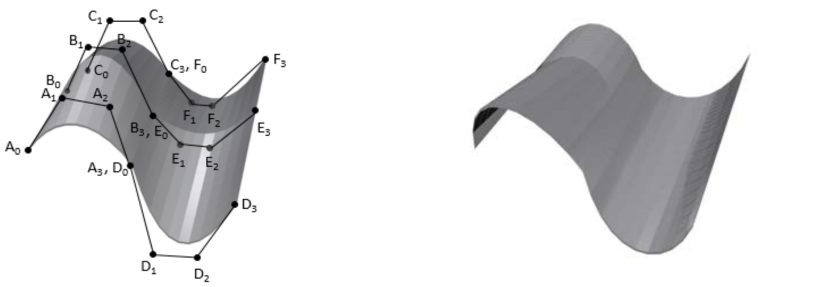
We next connect the developable surfaces in the direction of generating lines as shown in Fig. 6(a). The span lengths in the directions of directing curve and generating line are 60 m and 50 m, respectively. The directing curves are also defined by Bézier curves of order 3. The optimal shape is shown in Fig. 6.



(a) initial shape; $f(\mathbf{x}) = 324$ kNm (b) optimal shape; $f(\mathbf{x}) = 218$ kNm

Figure 6: Optimal developable surface connected in direction of direction lines.

Finally, four surfaces using Bézier curves of order 3 are connected in both directions of the direction of directing curves and generating lines as shown in Fig. 7(a). The span lengths in the directions of directing curve and generating line are 60 m and 50 m, respectively. The continuity conditions are given by assigning the quadrilaterals (A_2, A_3, E_0, E_1), (B_2, B_3, D_0, D_1), and (A_2, B_2, D_1, E_1) are in the same plane; (B_2, B_3, F_0, F_1), (C_2, C_3, E_0, E_1) and (B_2, C_3, E_1, F_1) are also in the same plane. Using these conditions, the optimal shape is obtained as shown in Fig. 7(b).



(a) initial shape; $f(\mathbf{x}) = 85.6$ kNm (b) optimal shape; $f(\mathbf{x}) = 47.9$ kNm

Fig. 7: Optimal developable surface connected in direction of direction lines.

6. Conclusions

A method has been presented for optimization of free-form shells as assemblage of developable surfaces. By assigning the developability conditions numerically, optimization process is simplified, and optimal shapes can be easily obtained using a nonlinear programming approach.

A complex surface can be modeled by assembling the developable surfaces. A simplified continuity condition has also been proposed so that ensure the continuity of normal, rather than enforcing continuity of tangent vector of directing curves.

The effectiveness of proposed approach has been demonstrated through numerical examples of surfaces modeled using Bézier curves of orders 2 and 3. It has been shown that the mechanical performance evaluated by the strain energy can be drastically improved by slight modification of surface shape.

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