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Series expansion method for determination of order of mechanism with partially rigid joints

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Abstract

A method for evaluating the order of mechanisms of partially rigid frames is presented. The frames are defined as assemblies of rigid beams and springs at their ends. Nonlinear compatibility conditions of translations and rotations are derived at member ends with respect to the nodal displacements and generalized strains representing rotations around arbitrarily inclined directions. The degree of kinematic indeterminacy is computed using singular value decomposition of the linear compatibility matrix. The order of mechanism is determined by the existence condition of higher-order coefficients of nodal displacements and generalized coordinates. The detailed procedure of the analysis is shown through the numerical examples of a two-bar and a four-bar linkages.

Keywords: mechanism, partially rigid joints, singular value decomposition, series expansion

1. Introduction

A mechanism that can have infinitesimal deformation without external load is called infinitesimal mechanism. By contrast, a mechanism is called finite mechanism if it can have large deformation without external load. Moreover, a mechanism has order, which is defined by the largest order of terms in power series expansion of displacements, in which strains vanish for all members (Salerno [1]). Generalized Maxwell's rule and the method using singular value decomposition (Liu *et al*. [2]) have been presented; however, they can determine only whether the order of the mechanism is first or higher.

Most of the previous studies considering the order of mechanisms focus on bar-joint systems consisting of bars connected by pin-joints. Additionally, their targets are mostly planar mechanisms. The stability of bar-joint systems is also studied in the field of prestressed structures (Ohsaki *et al*. [3], Zhang *et al*. [4]), structural rigidity (Connelly *et al*. [5]) and combinatorial rigidity (Katoh *et al*. [6]).

For the estimation of high-order stability of structures near the singular point of equilibrium under static loads, Koiter's asymptotic expansion method can be applied (Koiter [7], Thompson *et al*. [8]), and the method can also be used when singular points are overlapped (Ohsaki *et al*. [9]).

Salerno [1] determined the order of mechanisms of planar bar-joint frameworks by the condition that the coefficients of strains of the bars with respect to the path parameter become zero, and he showed that the method can also be applied when the number of the unstable modes is greater than one.

There exist some analytical approaches to evaluation of the order of 3-dimensional mechanisms of frames connected by revolute joints, for example, Bricard linkages, Goldberg linkages, etc. (Baker *et al*. [11]). Chen *et al*. [10] showed the existence of the bifurcation point on the deformation path of a

6R Bricard linkage. Numerical approach to order of 3-dimensional frames is difficult because of the non-commutativity of finite spatial rotations (Simo *et al*. [12], Hsiao *et al*. [13], Li [14]).

Partially rigid frames (Ohsaki *et al*. [17]) have revolute, screw, and/or universal joints freely rotating around only one or two axes, in contrast to truss structures consisting of bars connected by pin-joints rotatable around three axes. For application to retractable roof structures and deployable structures in outer space, etc., frame mechanisms should have small number of unstable modes so that the deformation process can be easily controlled.

In this study, we present a method for evaluating the order of deformation of partially rigid frame mechanisms consisting of rigid beam elements. Nonlinear compatibility conditions of translations and rotations are derived at member ends with respect to the nodal displacements and generalized strains.We show the detailed procedure through the numerical examples of a two-bar and a four-bar linkages (Guest *et al*. [18]).

2. Compatibility equations of partially rigid frames

2.1. Definition of rigid beam elements with generalized strains at member-ends

We consider partially rigid frames consisting of rigid beams and springs at the ends of the beams which deform by generalized strains between nodes and member-ends. Let K and M denotes the set of numbers of nodes and members. Nodes of two ends of the i th member are denoted as $k_{i1}, k_{i2} \in K$. Unit vector $\mathbf{t}_i^1$ is directed from node k_{i1} to k_{i2}, and we define unit vectors $\mathbf{t}_i^2$, $\mathbf{t}_i^3$ to satisfy $\mathbf{t}_i^1 \times \mathbf{t}_i^2 = \mathbf{t}_i^3$. Let $\mathbf{X}_k \in \mathbb{R}^3$ be the initial coordinate of node k, L_i be the length of member i. The vectors from the middle point to both ends connected to nodes k_{i1} and k_{i2} of member i are denoted by $\mathbf{r}_{ik_{i1}}$ and $\mathbf{r}_{ik_{i2}}$, respectively.

The translation vector of node k in the direction of global coordinates (x_1, x_2, x_3) is denoted by $\mathbf{U}_k = (U_k^1, U_k^2, U_k^3)^T \in \mathbb{R}^3$. The rotation vector of node k around axis x_k $(k = 1,2,3)$ is denoted by $\mathbf{\Theta}_k = (\Theta_k^1, \Theta_k^2, \Theta_k^3)^T \in \mathbb{R}^3$. The translation vector $\mathbf{V}_i = (V_i^1, V_i^2, V_i^3)^T \in \mathbb{R}^3$ of the middle point of member i is defined similarly.

Let $\mathbf{\Phi}_i = \phi_i \mathbf{n}_i$ denote a rotation vector of the middle point of member i with a unit vector $\mathbf{n}_i$.We define $\mathbf{t}_i^{l*}$ $(l = 1,2,3)$ by rotating $\mathbf{t}_i^l$ $(l = 1,2,3)$ around the axis $\mathbf{n}_i$ by the angle ϕ_i as follows (Cheng *et al*. [15]):

$$\mathbf{t}_i^{l*} = \mathbf{n}_i(\mathbf{n}_i \cdot \mathbf{t}_i^l) + [\mathbf{t}_i^l - \mathbf{n}_i(\mathbf{n}_i \cdot \mathbf{t}_i^l)]\cos\phi_i - (\mathbf{t}_i^l \times \mathbf{n}_i)\sin\phi_i \tag{1}$$

We define $\mathbf{r}_{ik_{i1}}^*$ and $\mathbf{r}_{ik_{i2}}^*$ by rotating $\mathbf{r}_{ik_{i1}}$ and $\mathbf{r}_{ik_{i2}}$, respectively, in the same manner.

Assuming that the middle point of member i and nodes k_{i1} and k_{i2} move independently, let $\Delta\tilde{\mathbf{U}}_{i1}, \Delta\tilde{\mathbf{U}}_{i2} \in \mathbb{R}^3$ and $\Delta\tilde{\mathbf{\Theta}}_{i1}, \Delta\tilde{\mathbf{\Theta}}_{i2} \in \mathbb{R}^3$ denote the translational and rotational incompatibility at two member-ends, respectively, as follows:

$$\Delta\tilde{\mathbf{U}}_{ij} = \mathbf{U}_{k_{ij}} - (\mathbf{V}_i + \mathbf{r}_{ik_{ij}}^*) + \mathbf{r}_{ik_{ij}} \quad (j = 1,2; i \in M) \tag{2}$$

$$\Delta\tilde{\mathbf{\Theta}}_{ij} = \mathbf{\Theta}_{k_{ij}} - \mathbf{\Phi}_i \quad (j = 1,2; i \in M) \tag{3}$$

The case for plane frame is illustrated in Fig. 1.

Let $\Delta\mathbf{U}_{i1}, \Delta\mathbf{U}_{i2}, \Delta\mathbf{\Theta}_{i1}, \Delta\mathbf{\Theta}_{i2} \in \mathbb{R}^3$ denote components of $\Delta\tilde{\mathbf{U}}_{i1}, \Delta\tilde{\mathbf{U}}_{i2}, \Delta\tilde{\mathbf{\Theta}}_{i1}, \Delta\tilde{\mathbf{\Theta}}_{i2}$, respectively, in the local coordinate system after deformation as

$$\Delta U_{ij}^{l} = \mathbf{t}_i^{l*} \cdot \Delta\tilde{\mathbf{U}}_{ij} \quad (l = 1, \cdots, 3;\ j = 1, 2;\ i \in M) \tag{4}$$

$$\Delta\Theta_{ij}^{l} = \mathbf{t}_i^{l*} \cdot \Delta\tilde{\mathbf{\Theta}}_{ij} \quad (l = 1, 2, 3;\ j = 1, 2;\ i \in M) \tag{5}$$

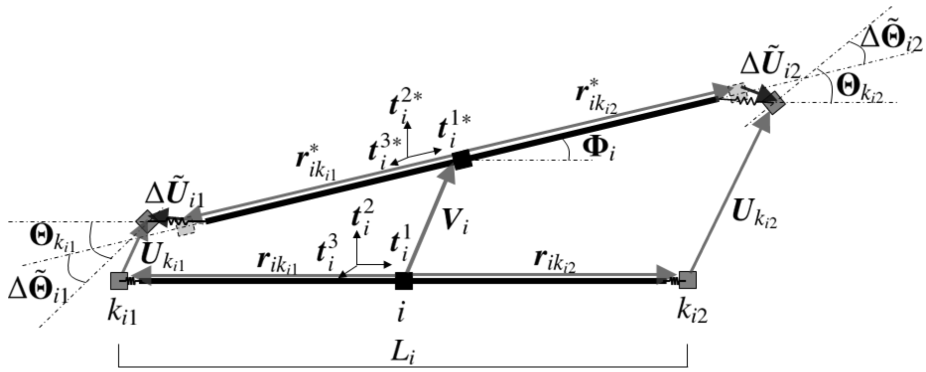


Figure 1: Translational and rotational incompatibility at two member-ends.

2.2. Rotational hinge at member-end

We add rotational degrees of freedom at member-ends, where inclined revolute joints are expected to exist (Tsuda *et al.* [19]). Between node k_{ij} and the spring of j th end of member i, we define a rotation vector $\boldsymbol{\theta}_{ij} = \theta_{ij}\mathbf{f}_{ij}^{*}$ where θ_{ij} is the angle and $\mathbf{f}_{ij}^{*} = f_{ij}^{1}\mathbf{t}_i^{1*} + f_{ij}^{2}\mathbf{t}_i^{2*} + f_{ij}^{3}\mathbf{t}_i^{3*}$ is the unit directional vector of the axis of rotation. Now the rotational incompatibility, which is regarded as generalized strain, in the spring is formulated as follows:

$$\Delta\hat{\mathbf{\Theta}}_{ij} = \Delta\mathbf{\Theta}_{ij} + \boldsymbol{\theta}_{ij} \tag{6}$$

$$\Delta\hat{\Theta}_{ij}^{l} = \Delta\Theta_{ij}^{l} + \theta_{ij} f_{ij}^{l} \tag{7}$$

When we have h hinges, additional variables are represented by $\boldsymbol{\theta} \in \mathbb{R}^h$, which consists of θ_{ij}.

We merge ΔU_{ij}^{l} in (4) and $\Delta\Theta_{ij}^{l}$ in (7) into $\mathbf{G} \in \mathbb{R}^m$, which we call generalized strain vector. Similarly we define displacement vector $\mathbf{W} \in \mathbb{R}^n$ as an assemblage of $\mathbf{U}, \mathbf{\Theta}, \mathbf{V}, \mathbf{\Phi}$ and $\boldsymbol{\theta}$. Let m_0, n_0 and h represent the numbers of members, nodes and constrained degrees of freedom, respectively. Then, the numbers of components of $\mathbf{G}$ and $\mathbf{W}$, denoted by m and n, respectively, are determined as $m = 12m_0$ and $n = 6n_0 + 6m_0 + h - s$.

3. Derivation of infinitesimal mechanism modes

We parameterize $\mathbf{W}$ in terms of a path parameter ξ. In the following, we use $(\,)'$ as the derivative with respect to ξ and adopt Einstein's summation convention when an index is repeated in one term.

If there exist any mechanisms of a frame, the generalized strain must remain zero along them; i.e.,

$$\mathbf{G}(\mathbf{W}(\xi)) = \mathbf{0} \tag{8}$$

Differentiating (8) with respect to ξ and substituting $\xi = 0$, we obtain

$$\mathbf{\Gamma W}' = \mathbf{0} \tag{9}$$

where $\mathbf{\Gamma} \in \mathbb{R}^{m \times n}$ is a constant matrix of which the (s,i) element is $\partial G_s / \partial W_i \big|_{\xi=0}$. If any non-zero $\mathbf{W}'$ satisfying (9) is found, we determine that the frame has at least first order (i.e. infinitesimal) mechanism.

$\mathbf{\Gamma}$ is rewritten, as follows, using singular value decomposition (SVD):

$$\mathbf{\Gamma} = \mathbf{B \Sigma H}^T \tag{10}$$

Defining $r = \mathrm{rank}(\mathbf{\Gamma})$, $p = n - r$ and $q = m - r$, we can express matrices $\mathbf{\Gamma}$, $\mathbf{H}$ and $\mathbf{B}$ as

$$\mathbf{\Sigma} = \begin{bmatrix} \mathrm{diag}\left(\lambda_1, \cdots, \lambda_r\right) & \mathbf{O}_{r \times p} \\ \mathbf{O}_{q \times r} & \mathbf{O}_{q \times p} \end{bmatrix} \in \mathbb{R}^{m \times n} \tag{11}$$

$$\mathbf{H} = \left[\boldsymbol{\eta}_1, \quad \cdots, \quad \boldsymbol{\eta}_n\right] \in \mathbb{R}^{n \times n} \tag{12}$$

$$\mathbf{B} = \left[\boldsymbol{\beta}_1, \quad \cdots, \quad \boldsymbol{\beta}_m\right] \in \mathbb{R}^{m \times m} \tag{13}$$

where the singular values are ordered as $\lambda_1 \geq \cdots \geq \lambda_r > 0$, and $\mathbf{H}$ and $\mathbf{B}$ are orthogonal matrices consisting of singular vectors.

Let $\mathrm{im}(\mathbf{\Gamma})$ and $\ker(\mathbf{\Gamma})$ be the column space and the null space (Meyer [16]) of $\mathbf{\Gamma}$. Then we can show that $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_r$ are the bases of $\mathrm{im}(\mathbf{\Gamma})$, $\boldsymbol{\eta}_{r+1}, \cdots, \boldsymbol{\eta}_{r+p}$ are the bases of $\ker(\mathbf{\Gamma})$, $\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_r$ are the bases of the row space $\mathrm{im}(\mathbf{\Gamma}^T)$, and $\boldsymbol{\beta}_{r+1}, \cdots, \boldsymbol{\beta}_{r+q}$ are the bases of the left null space $\ker(\mathbf{\Gamma}^T)$.

When $\mathbf{W}'$ is an infinitesimal mechanism, $\mathbf{W}'$ is written as the linear combination of the bases of $\ker(\mathbf{\Gamma})$. Therefore, $\boldsymbol{\eta}_{r+1}, \cdots, \boldsymbol{\eta}_{r+p}$ represent the infinitesimal mechanism modes, and p is the number of mechanism modes. Similarly, we can show that the self-equilibrium force vector $\mathbf{F}$ is written as the linear combination of the bases $\boldsymbol{\beta}_{r+1}, \cdots, \boldsymbol{\beta}_{r+q}$ of $\ker(\mathbf{\Gamma}^T)$, where q is the number of self-equilibrium modes.

For simplicity, hereinafter we reorder the singular values and vectors conversely; i.e., we rename $\boldsymbol{\eta}_n, \cdots, \boldsymbol{\eta}_1$ as $\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_n$, $\boldsymbol{\beta}_m, \cdots, \boldsymbol{\beta}_1$ as $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_m$ and $\lambda_1, \cdots, \lambda_r$ as $\lambda_r, \cdots, \lambda_1$.

4. Mechanism including higher-order terms

Assuming that a frame has a single infinitesimal mechanism mode, we investigate whether the mechanism has higher order terms or not. We define $\mathbf{W}(\xi)$ as

$$\mathbf{W} = \xi \boldsymbol{\eta}_1 + \alpha_2 \boldsymbol{\eta}_2 + \cdots + \alpha_n \boldsymbol{\eta}_n \tag{14}$$

where $\boldsymbol{\eta}_1$ is a mechanism mode and $\alpha_j(\xi)$ $(j = 2, \cdots, n)$ are the coefficients of other basis vectors with respect to ξ. Pre-multiplying $\boldsymbol{\eta}_1^T$ to the both sides of (16), we have $\boldsymbol{\eta}_1^T \mathbf{W} = \xi$. Differentiating it with respect to ξ, we obtain $\boldsymbol{\eta}_1^T \mathbf{W}' = 1$ and $\boldsymbol{\eta}_1^T \mathbf{W}'' = \boldsymbol{\eta}_1^T \mathbf{W}''' = \cdots = 0$.

Assuming the infinitesimal mechanism $\mathbf{W}' = \boldsymbol{\eta}_1$ is obtained, we investigate the condition for existence of second order mechanism. Differentiating $\dfrac{\partial G_s}{\partial W_i} = 0$ $(s = 1, \cdots, m)$ with respect to ξ, we obtain

$$\frac{\partial^2 G_s}{\partial W_i \partial W_j} W_i' W_j' + \frac{\partial G_s}{\partial W_i} W_i'' = 0 \ \ (s = 1, \cdots, m) \tag{15}$$

From (15), $\mathbf{W}''$ is determined as the solution of following linear equation.

$$\mathbf{\Gamma W}'' = \mathbf{g}^{(2)} \tag{16}$$

$$g_s^{(2)} = -\frac{\partial^2 G_s}{\partial W_i \partial W_j} W_i' W_j' \, (s = 1, \cdots, m) \tag{17}$$

where $\mathbf{g}^{(2)} \in \mathbb{R}^m$ is a constant vector calculated from $\mathbf{W}'$. If there exists $\mathbf{W}''$ satisfying (16), the frame has an at least second order mechanism. Note that (16) has the solution $\mathbf{W}''$ if and only if $\mathbf{g}^{(2)} \in \mathrm{im}(\mathbf{\Gamma})$; i.e., $\mathbf{g}^{(2)}$ is orthogonal to all bases of $\ker(\mathbf{\Gamma}^T)$. We summarize the conditions for existence of $\mathbf{W}''$ for the cases $q = 0$ and $q \geq 1$, respectively, as follows:

(a) $q = 0$ In this case, $\ker(\mathbf{\Gamma}^T)$ has no basis and (20) always has the solution.

(b) $q \geq 1$ The bases of $\ker(\mathbf{\Gamma}^T)$ are self-equilibrium force modes $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_q$; thus $\mathbf{W}''$ exists when the following equations hold:

$$\boldsymbol{\beta}_i^T \mathbf{g}^{(2)} = 0 \, (i = 1, \cdots, q) \tag{18}$$

Note that we can regard $\mathbf{g}^{(2)}$ as a high-order strain vector generated by $\mathbf{W}'$. Therefore, (18) indicates that the works done by the forces $\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_q$ against the strain $\mathbf{g}^{(2)}$ vanish.

When the conditions (a) or (b) are satisfied, we can obtain $\mathbf{W}''$ from (16). Therefore, each of (a) and (b) is the sufficient condition for existence of second order mechanism. The conditions of third and higher orders can be derived in the same manner.

Finally, the deformation including higher order terms can be expressed as

$$\mathbf{W} = \xi \mathbf{W}' + \frac{1}{2!} \xi^2 \mathbf{W}'' + \frac{1}{3!} \xi^3 \mathbf{W}''' + \cdots \tag{19}$$

We can successively determine the terms up to arbitrary orders in (19), if the mechanism evaluated above is a finite mechanism.

5. Numerical examples

5.1. Example 1: two-bar linkages

We analyze two models A and B shown in Figure 2. In both models, two members having the same length are connected at node 2, all translational and rotational components except rotation around Y-axis are constrained at node 1, and all translational components except X-directional displacement are constrained at node 3. Furthermore, both models have a hinge at the end of member 2 connected to node 2. The difference of two models is the direction of the hinge; the axis of the hinge of model A is parallel to Y-axis, while the axis of the hinge of model B is inclined 45 degrees from X-axis and Y-axis.

In both of the two models, $m_0 = 2$, $n_0 = 3$, $h = 1$, $s = 7$, $m = 24$, $n = 24$ and $r = \mathrm{rank}(\mathbf{\Gamma}) = 23$. Thus, they both have one infinitesimal mechanism mode and one self-equilibrium force mode, because $p = q = 1$. Evaluating the second order condition (18), we find that $\boldsymbol{\beta}_1{}^T \mathbf{g}^{(2)} = 0.0000$ is satisfied for model A, while $\boldsymbol{\beta}_1{}^T \mathbf{g}^{(2)} = -0.0843 \neq 0$ for model B. Consequently, we can determine that the mechanism of model A is at least second order and the mechanism of model B is first order (infinitesimal).

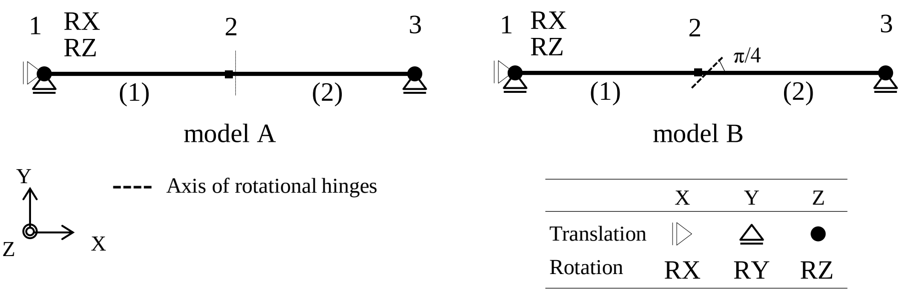


Figure 2: Two-bar linkages.

5.2. Example 2: four-bar linkage

Next, we analyze a square model on XY-plane as shown in Figure 3, which has four members connected at four nodes. Each member has hinges at both ends, and the axes of the hinges are defined in the initial state in the global coordinate system as follows:

$$\mathbf{f}_{11} = \mathbf{f}_{44} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{f}_{12} = \mathbf{f}_{21} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \mathbf{f}_{22} = \mathbf{f}_{31} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{f}_{32} = \mathbf{f}_{41} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \tag{20}$$

The constraint condition of the nodes is shown in Figure 3, where $m_0 = 4$, $n_0 = 4$, $h = 8$ and $s = 14$; thus, the numbers of rows and columns of matrix $\boldsymbol{\Gamma}$ are $m = 48$ and $n = 42$, respectively. From the SVD of $\boldsymbol{\Gamma}$, we obtain $r = 41$; hence. $p = n - r = 1$ and $q = m - r = 7$ are determined. Therefore, the model has one infinitesimal mechanism mode and seven self-equilibrium force modes.

We confirmed all of the seven equations $\boldsymbol{\beta}_j{}^T \mathbf{g}^{(2)} = 0$ ($j = 1, \cdots, 7$) of the second order condition (18) are satisfied to obtain non-zero solution of $\mathbf{W}''$. We can find that the equations of the third order condition $\boldsymbol{\beta}_j{}^T \mathbf{g}^{(3)} = 0$ ($j = 1, \cdots, 7$) are also satisfied, and we can obtain $\mathbf{W}'''$ in the same manner. Vectors $\mathbf{W}'$, $\mathbf{W}''$ and $\mathbf{W}'''$ are shown in Figure 4, where their scales are arbitrary. The mechanism $\mathbf{W}$ determined by (19) considering up to third order terms are shown in Figure 5, where ξ is varied from 0 to 2 by increment $\Delta\xi = 0.5$.

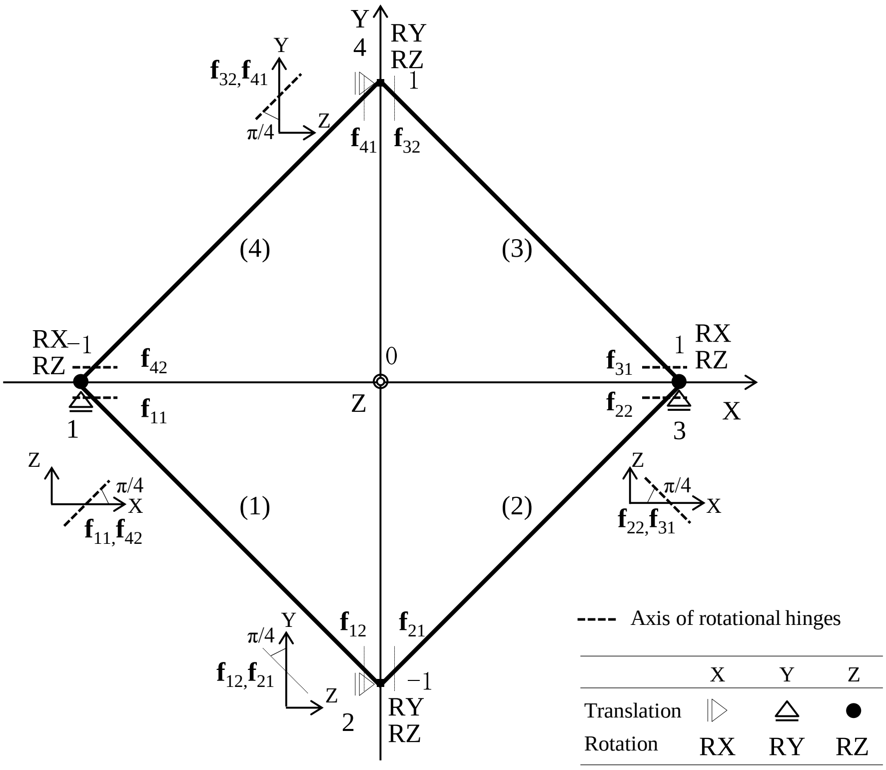
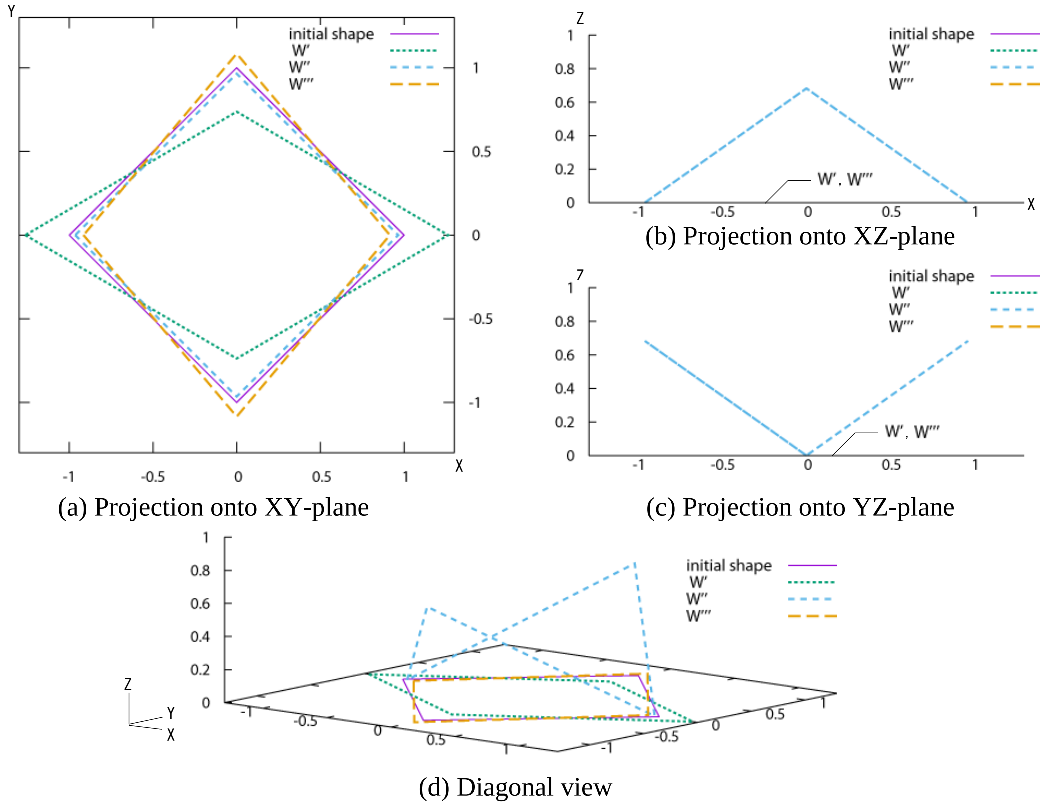


Figure 3: Four-bar linkage.



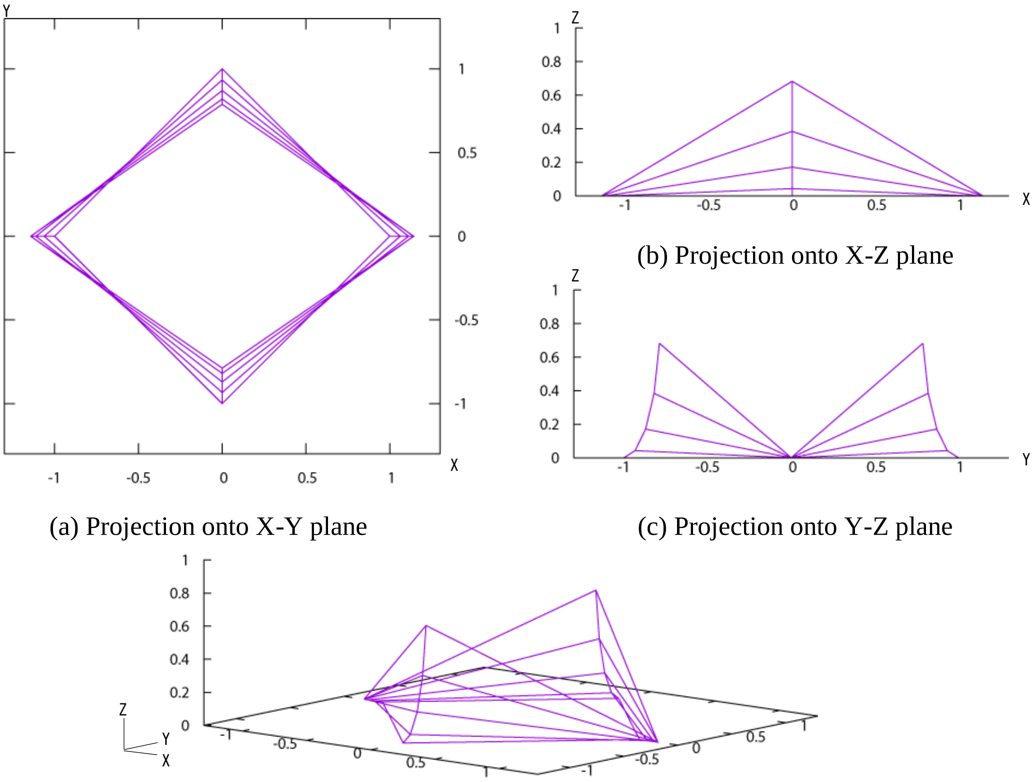
(a) Projection onto XY-plane

(b) Projection onto XZ-plane

(c) Projection onto YZ-plane

(d) Diagonal view

Figure 4: $\mathbf{W}', \mathbf{W}'', \mathbf{W}'''$ of Figure 3.



(a) Projection onto X-Y plane

(b) Projection onto X-Z plane

(c) Projection onto Y-Z plane

(d) Diagonal view

Figure 5: Third order mechanism of Figure 3.

6. Conclusion

We have presented a method for determining the order of mechanisms of partially rigid frames by defining them as assemblies of rigid beams and springs at their ends. The incompatiblility at the member ends is regarded as generalized strain. By successively differentiating the generalized strain with respect to the path parameter, the conditions for existence of higher order terms of the mechanisms are derived.

The first numerical example has shown that the method presented here can make a distinction between two mechanisms of the frame with different hinge directions, which cannot be distinguished on infinitesimal deformation theory. We also have shown in the second numerical example that our method may be applied to frames with many members.

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