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A Sequential Second-Order Cone Programming for Stability Analysis of Nonsmooth Mechanical Systems

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1. Abstract
This paper discusses an optimization-based algorithm for the stability determination of a given equilibrium point of a statically-loaded finite-dimensional nonsmooth mechanical system with unilateral constraints. As for situations of unilateral constraints we focus on (i) cable structures, (ii) frictionless contacts, and (iii) elastic-plastic structures. It is shown that the stability of a given equilibrium point of such a mechanical system can be determined from the optimal value of a maximization problem of a convex function over a convex set. We propose an algorithm for the stability determination problem, in which second-order cone programming problems are solved sequentially.

2. Keywords: Second-order cone program; Nonsmooth mechanics; Stability analysis; Frictionless contact; Unilateral constraint.

3. Introduction
In this paper we propose a numerical method for stability determination of a given equilibrium point of a statically-loaded finite-dimensional mechanical system, when the displacements and/or stresses are subjected to unilateral constraints. Since the stability of the static equilibrium point is an important issue of nonlinear mechanics, various criteria on the stability of equilibrium points have been proposed for elastic and inelastic structures [4, 23].

It is known that various classes of structural systems are governed by the unilateral constraints on displacements and/or stresses; see, e.g. Duvaut and Lions [6]. This paper deals with the following problems in nonsmooth mechanics that share the common mathematical framework:

(i) Structures including no-compression cables;

(ii) Frictionless contacts;

(iii) Elastic-plastic trusses with nonnegative hardening.

A cable member cannot transmit compressive force. This property is referred to as the *stress unilateral behavior*, and was studied by Panagiotopoulos [17] by means of the variational inequality. Later, the existence and uniqueness of static equilibrium solutions were investigated for cable networks in [3, 9, 27] and for prestressed pin-jointed structural systems in [15, 16].

For frictionless contacts, numerical path-tracing methods still receive much attention [8, 26]. The stability in frictionless contacts was investigated theoretically by Klarbring [12] and the references therein. Tschöpe *et al.* [24] proposed a numerical method for finding limit points in frictionless contacts. We attempt in this paper to determine the stability of an equilibrium point which is given *a priori*.

Elastic-plastic analysis can be regarded as a problem in which stresses are subjected to the unilateral constraints [21], if infinitesimal displacements are considered. Hill [7] derived a sufficient condition for stability of a given equilibrium point of elastic-plastic structures. For elastic-plastic problems, we restrict ourselves to the directional stability of truss structures for simplicity.

This paper presents a numerical method to determine the stability of the given equilibrium point of a structures subjected to unilateral constraints. It is emphasized that the method should be applicable to large-scale problems, particularly, problems with a large number of unilateral constraints. We achieve this aim by using numerical optimization. For frictional contact problems, a mathematical programming approach was proposed to find the directionally unstable points by Pinto da Costa *et al.* [19], which is based on an enumeration of solutions to the complementarity problem. Casciaro and Mancusi [5] performed the imperfection sensitivity analysis numerically by solving a nonconvex quadratic programming problem.

We present a unified perspective, as well as general formulations, for the stability determination of the structural systems (i)–(iii). For a given equilibrium point, we shall show that the (directional) stability of mechanical systems belonging to these three classes can be determined by solving an optimization problem, specifically, the maximization problem of a convex quadratic function over a convex homogeneous quadratic inequality and some linear inequalities.

We next propose a solution technique for the stability determination problem. Since the problem is nonconvex, the conventional local method may converge to a local solution, which implies that the stability cannot be determined correctly. On the other hand, any global optimization method seems to be too expensive for mechanical engineers from the view point of computational cost. From a practical point of view, it seems that a global optimization approach is not suitable for stability analysis. Hence, in this paper, we choose a local method, which quite often converges to the global optimal solution.

We show that the stability determination problem presented can be embedded into the form of the *DC* (difference of convex functions) *program* [2]. The DC algorithm was proposed as a local method solving the DC programming problem, and has been examined by various kinds of DC programming problems; see the review paper [2] and the references therein. It has been observed that the DC algorithm quite often converges to global optimal solution, although the convergence to the global solution is not guaranteed theoretically. Based on the DC algorithm we propose an algorithm for the stability determination problem, At each iteration of which we solve a *second-order cone programming* (SOCP) problem [1] by using the primal-dial interior-point method.

4. General framework for stability analysis

4.1. Stability criterion

Consider a finite-dimensional structure in two- or three-dimensional space. The structure is subjected to static nodal loads. Let $\boldsymbol{\xi}^0 \in \mathbb{R}^k$ denote the vector of state variables describing the static equilibrium point, which consists of the total nodal displacement vector and the generalized stress vector. Suppose that we are given the equilibrium point $\boldsymbol{\xi}^0$ under the specified external load. Let n^{d} denote the number of degrees of freedom, and $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$ denotes the vector of infinitesimal incremental displacements from the equilibrium point $\boldsymbol{\xi}^0$. We denote by $\mathcal{A}(\boldsymbol{\xi}^0) \subseteq \mathbb{R}^{n^{\mathrm{d}}}$ the set of all admissible incremental displacements $\boldsymbol{u}$ satisfying the boundary conditions.

4.1.1. Stability condition for elastic structures

For elastic structures subjected to unilateral stress and/or displacement constraints, the total potential energy $\Pi(\boldsymbol{u})$ is defined for admissible incremental displacements vector $\boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0)$. By application of Liapunov's direct method [13], a sufficient condition of stability is given as follows.

Proposition 4.1. *The equilibrium point* $\boldsymbol{\xi}^0$ *is stable if* $\Pi : \boldsymbol{u} \mapsto \Pi(\boldsymbol{u})$ *is continuously differentiable at any* $\boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0)$ *and if* Π *has an isolated minimum at* $\boldsymbol{u} = \boldsymbol{0}$.

Let $\boldsymbol{K}(\boldsymbol{u}; \boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\mathrm{d}}}$ denote the tangential stiffness matrix, $\mathcal{S}^{n^{\mathrm{d}}}$ is the set of $n^{\mathrm{d}} \times n^{\mathrm{d}}$ real symmetric matrices. We denote by $v(\boldsymbol{u})$ the (twice of) second-order term of the increment of the potential energy corresponding to $\boldsymbol{u}$ at $\boldsymbol{\xi}^0$, which is given by

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{\xi}^0) \boldsymbol{u}. \tag{1}$$

Define $v^* \in \mathbb{R}$ by

$$v^* := \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0),\ \|\boldsymbol{u}\| = 1 \right\}, \tag{2}$$

where $\|\boldsymbol{u}\| = (\boldsymbol{u}^{\mathrm{T}} \boldsymbol{u})^{1/2}$. Since the stability can be determined only by the sign of v^*, we impose the normalization condition on $\boldsymbol{u}$ in (2) without loss of generality.

The sufficient conditions for stability and instability are then given as follows.

Proposition 4.2. *The equilibrium point* $\boldsymbol{\xi}^0$ *is stable (resp. unstable) if* $v^* > 0$ *(resp.* $v^* < 0$*).*

Note that Proposition 4.2 gives a stability criterion for frictionless contact problems (see, e.g., [11] for more details) and elastic structures including no-compression cables.

4.1.2. Stability condition for elastic-plastic structures
Since the total potential energy cannot be defined for elastic-plastic structures, we employ the notion of *directional instability* [4] which is defined by using v^* as follows.

Definition 4.3. The equilibrium point $\boldsymbol{\xi}^0$ is said to be directionally stable (resp. directionally unstable) if $v^* > 0$ (resp. $v^* < 0$). The equilibrium point is unstable if it is directionally unstable.

Note that the directional stability is a necessary condition for the stability of an elastic-plastic structure [4]. However, we restrict ourselves to the directional stability of elastic-plastic structure. In the remainder of the paper, we omit the term *directional* as far as no confusion is possible.

4.1.3. Stability determination problem
It follows from sections 4.1.1 and 4.1.2 that the stability of both elastic and elastic-plastic systems are determined by finding a global optimal solution of the problem (2). We focus on a case in which the matrix $\boldsymbol{K}(\boldsymbol{u};\boldsymbol{\xi}^0)$ is indefinite, even for the fixed $\boldsymbol{u}$.

4.2. Reduction to feasibility problem
In sections 5–7, we show that v^* defined by (2) can be obtained as the optimal value of an optimization problem in the form of

$$v^* = \min_{\boldsymbol{u},\boldsymbol{z}} \left\{ \boldsymbol{u}^{\mathrm{T}} \boldsymbol{Q}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z} : (\boldsymbol{u},\boldsymbol{z}) \in \mathcal{F},\ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{3}$$

See Proposition 5.2 and the problem (30) for the detail of each specific problem. Note that $\boldsymbol{Q}_0 \in \mathcal{S}^{n^{\mathrm{d}}}$ and $\boldsymbol{Q}_1 \in \mathcal{S}^m_{++}$ are constant in (3), where $\mathcal{S}^n_{++}$ is the set of $n \times n$ real symmetric positive definite matrices. Moreover, $\mathcal{F} \subseteq \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m$ is a convex set that can be represented in the form of

$$\mathcal{F} = \left\{ (\boldsymbol{u},\boldsymbol{z}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \mid \boldsymbol{A}_u \boldsymbol{u} + \boldsymbol{A}_z \boldsymbol{z} \geq \boldsymbol{0} \right\}, \tag{4}$$

where $\boldsymbol{A}_u$ and $\boldsymbol{A}_z$ are constant matrices with appropriate sizes. The stability of $\boldsymbol{\xi}^0$ is determined by solving (3) instead of (2).

Let $\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_i) \in \mathbb{R}^n \mid x_i > 0 \ (i = 1, \dots, n)\}$. Let $\widetilde{\lambda} \in \mathbb{R}_{++}$ be a constant satisfying

$$\boldsymbol{Q}_0 + \widetilde{\lambda} \boldsymbol{I} \in \mathcal{S}^n_{++}, \tag{5}$$

i.e. $\widetilde{\lambda}$ is greater than the modulus of the smallest eigenvalue of $\boldsymbol{Q}_0$. Define $\widetilde{\boldsymbol{Q}}_0 \in \mathcal{S}^{n^{\mathrm{d}}}_{++}$ by

$$\widetilde{\boldsymbol{Q}}_0 = \boldsymbol{Q}_0 + \widetilde{\lambda} \boldsymbol{I}. \tag{6}$$

For simplicity, we use the following notations:

$$\boldsymbol{x} := \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix}, \quad \widetilde{f}(\boldsymbol{u},\boldsymbol{z}) = \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{Q}}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z}, \quad g(\boldsymbol{u},\boldsymbol{z}) = \|\boldsymbol{u}\|^2 - 1. \tag{7}$$

Define $\widetilde{v}$ by

$$\begin{aligned} \widetilde{v} &= \min_{\boldsymbol{x}} \left\{ \widetilde{f}(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{F},\ g(\boldsymbol{x}) \geq 0 \right\} \\ &= \min_{\boldsymbol{u},\boldsymbol{z}} \left\{ \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{Q}}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z} : (\boldsymbol{u},\boldsymbol{z}) \in \mathcal{F},\ \|\boldsymbol{u}\|^2 - 1 \geq 0 \right\}. \end{aligned} \tag{8}$$

The relation between (3) and (8) is stated as follows.

Proposition 4.4. *The problems* (3) *and* (8) *share the same set of optimal solutions, and* $\widetilde{v} = v^* + \widetilde{\lambda}$.

The following is an immediate corollary of Definition 4.3 and Proposition 4.4:

Corollary 4.5. *The equilibrium point* $\boldsymbol{\xi}^0$ *is stable (resp. unstable) if* $\widetilde{v} > \widetilde{\lambda}$ *(resp.* $\widetilde{v} < \widetilde{\lambda}$*).*

We next propose an algorithm for (8) which for most cases converges to the global optimal solution within the computational time acceptable for real engineering applications. By exchanging the objective function and the nonconvex constraint of (8), we consider the following family of problems with respect to a parameter λ:

$$g^*(\lambda) := \max_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \leq \lambda \right\}, \tag{9}$$

which defines the function $g^* : \mathbb{R}_{++} \to \mathbb{R}$. Note that (9) is a maximization of the convex function over the convex set.

The relation investigated in Proposition 4.6 is sometimes called *duality* between the objective and constraint functions [25].

Proposition 4.6. *For any* $\lambda \in \mathbb{R}_{++}$*,*

(i) $\widetilde{v} \geq \lambda$ *implies* $g^*(\lambda) \leq 0$.

(ii) $g^*(\lambda) \leq 0$ *implies* $\widetilde{v} \geq \lambda$.

The following is the key result of this paper, which implies that the stability can be determined by solving the problem (9) instead of the problem (8).

Theorem 4.7. *The equilibrium point* $\boldsymbol{\xi}^0$ *is*

(i) *stable if* $g^*(\widetilde{\lambda}) < 0$*;*

(ii) *unstable if* $g^*(\widetilde{\lambda}) > 0$.

Furthermore,

(iii) $\widetilde{v} = \lambda$ *if and only if* $g^*(\lambda) = 0$.

It is clear that a feasible solution $\overline{\boldsymbol{x}}$ of (8) is optimal if and only if $\widetilde{f}(\overline{\boldsymbol{x}}) = \widetilde{v}$. Hence, the following is an immediate corollary of Theorem 4.7 (iii).

Corollary 4.8. *A feasible solution* $\overline{\boldsymbol{x}}$ *of the problem* (8) *is optimal if and only if* $g^*(\widetilde{f}(\overline{\boldsymbol{x}})) = 0$.

4.3. Sequential convex optimization algorithm

We have seen in Theorem 4.7 that the stability of the given equilibrium point is determined by solving the problem (9). In this section we first propose an algorithm for (9).

We first show that the problem (9) can be reformulated as a *DC* (difference of convex functions) *programming* problem [2]. Let $\rho \in \mathbb{R}_{++}$ be a constant. Define $h_1,\ h_2 : \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \to \mathbb{R}$ by

$$h_1(\boldsymbol{x}) = \frac{\rho}{2}\|\boldsymbol{x}\|^2 + g(\boldsymbol{x}), \quad h_2(\boldsymbol{x}) = \frac{\rho}{2}\|\boldsymbol{x}\|^2. \tag{10}$$

Note that h_1 and h_2 are strictly convex. Then the problem (9) is equivalently rewritten as

$$\max_{\boldsymbol{x}} \left\{ h_1(\boldsymbol{x}) - h_2(\boldsymbol{x}) : \boldsymbol{x} \in \widetilde{\mathcal{F}}(\lambda) \right\}, \tag{11}$$

which is a DC programming problem.

The DC algorithm generates $\{\boldsymbol{x}^k\}$ by defining $\boldsymbol{x}^{k+1}$ as the solution to the following problem [2]

$$\max_{\boldsymbol{x}} \left\{ \left[(\boldsymbol{y}^k)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^k) + h_1(\boldsymbol{x}^k) \right] - h_2(\boldsymbol{x}) \mid \boldsymbol{x} \in \widetilde{\mathcal{F}}(\lambda) \right\}, \tag{12}$$

where the dual variables $\boldsymbol{y} \in \mathbb{R}^{n^{\mathrm{d}} + n^{\mathrm{m}}}$ is updated by

$$\boldsymbol{y}^k := \rho \boldsymbol{x}^k + \nabla g(\boldsymbol{x}^k). \tag{13}$$

Substitution of (13) into (12) yields

$$\max_{\boldsymbol{x}} \left\{ (\rho \boldsymbol{x}^k + \nabla g(\boldsymbol{x}^k))^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^k) + \left(\frac{\rho}{2}\|\boldsymbol{x}^k\|^2 + g(\boldsymbol{x}^k) \right) - \frac{\rho}{2}\|\boldsymbol{x}\|^2 : \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \leq \lambda \right\}. \tag{14}$$

By multiplying a constant $2/\rho$ and eliminating the constant terms, the objective function of (14) is simplified without changing the optimal solution as

$$\max_{\boldsymbol{u},\boldsymbol{z}} \left\{ -\left\|\boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k\right\|^2 - \left\|\boldsymbol{z} - \boldsymbol{z}^k\right\|^2 : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{F},\ \widetilde{f}(\boldsymbol{u}, \boldsymbol{z}) \leq \lambda \right\}. \tag{15}$$

Note that (15) is an SOCP (second-order cone programming) problem [1], which can be solved efficiently by using the primal-dual interior-point method. The following algorithm solves a convex problem (15) sequentially to obtain a solution of (9).

Algorithm 4.9 (sequential convex optimization method for (9)).

Step 0: Choose $(\boldsymbol{u}^0, \boldsymbol{z}^0) \in \mathbb{R}^{n^{\rm d}} \times \mathbb{R}^m$, $\rho > 0$, and the tolerance $\epsilon > 0$. Set $k := 0$.

Step 1: Find the optimal solution $(\boldsymbol{u}^{k+1}, \boldsymbol{z}^{k+1})$ of (15).

Step 2: If $\|(\boldsymbol{u}^{k+1}, \boldsymbol{z}^{k+1}) - (\boldsymbol{u}^k, \boldsymbol{z}^k)\| \le \epsilon$, then stop. Otherwise, set $k \leftarrow k+1$, and go to Step 1.

Algorithm 4.9 is guaranteed to be well-defined by the following proposition in the sense that the subproblem (15) solved at each iteration has the unique solution. From Theorem 4.7 we obtain the following corollary, which gives a sufficient condition for instability.

Corollary 4.10. *Put $\lambda := \widetilde{\lambda}$ in the problem (9), and let $\boldsymbol{x}^* = (\boldsymbol{u}^*, \boldsymbol{z}^*)$ be an accumulation point of a sequence $\{\boldsymbol{x}^k\}$ generated by Algorithm 4.9. If $g(\boldsymbol{x}^*) > 0$, then the equilibrium point $\boldsymbol{\xi}^0$ is unstable.*

Since Algorithm 4.9 is based on a local optimality condition of the problem (9), it cannot guarantee the global optimality of a solution obtained. However, it has been observed (see, e.g. [2]) that the DC algorithm very often converges to global optimal solutions of various nonconvex optimization problems in practice. Therefore, from Theorem 4.7 and the fact that Algorithm 4.9 with $\lambda := \widetilde{\lambda}$ provides a lower bound of $g^*(\widetilde{\lambda})$, the equilibrium point is stable for most cases if $g(\boldsymbol{x}^*) < 0$.

Note that, from Theorem 4.7, it is sufficient to compute $g^*(\widetilde{\lambda})$ in order to determine the stability. On the contrary, when we want to know the incremental displacement corresponding to the minimum increment of potential energy, the problem (8) is to be solved. It follows from Corollary 4.8 that (8) can be solved by using a bi-section method as follows.

Algorithm 4.11 (bisection method for (8)).

Step 0: Choose $\underline{\lambda}^0$ and $\overline{\lambda}^0$ satisfying $0 < \underline{\lambda}^0 \le \lambda^* \le \overline{\lambda}^0$, and the tolerance $\epsilon > 0$. Set $k := 0$.

Step 1: If $\overline{\lambda}^k - \underline{\lambda}^k \le \epsilon$, then stop. Otherwise, set $\lambda := (\underline{\lambda}^k + \overline{\lambda}^k)/2$.

Step 2: Find an optimal solution $(\boldsymbol{u}^*, \boldsymbol{z}^*)$ of the problem (9) by using Algorithm 4.9.

Step 3: If $g(\boldsymbol{x}^*) < 0$, then set $\underline{\lambda}^{k+1} := \lambda$ and $\overline{\lambda}^{k+1} := \overline{\lambda}^k$. Otherwise, set $\overline{\lambda}^{k+1} := \lambda$ and $\underline{\lambda}^{k+1} := \underline{\lambda}^k$.

Step 4: Set $k := k+1$, and go to Step 1.

5. Structures including no-compression cables

Consider an elastic finite dimensional structure containing cable members that cannot transmit compressive forces, i.e. the cable member is assumed to consist of *no-compression material*. The equilibrium point $\boldsymbol{\xi}^0$ is defined by the total displacement vector in this section. The compatibility condition at $\boldsymbol{\xi}^0$ between the incremental member elongation c_j and the incremental displacements $\boldsymbol{u}$ is written as

$$\boldsymbol{c}(\boldsymbol{u}) = \boldsymbol{B}(\boldsymbol{\xi}^0)^{\rm T}\boldsymbol{u}, \tag{16}$$

where $\boldsymbol{B}(\boldsymbol{\xi}^0) \in \mathbb{R}^{n^{\rm m} \times n^{\rm d}}$ is a constant matrix, $n^{\rm m}$ is the number of members, and $n^{\rm d}$ is the number of degrees of freedom of displacements.

5.1. Stability of structures with cables

Let $\mathcal{J} \subseteq \{1, \dots, n^{\rm m}\}$ denote the set of all indices of cable members with vanishing elongations at $\boldsymbol{\xi}^0$. We denote by $k_j^{\rm c}(c_j)$ the elongation stiffness at $\boldsymbol{\xi}^0$ given by

$$k_j^{\rm c}(c_j) = \begin{cases} d_j & \text{if } c_j \ge 0, \\ 0 & \text{if } c_j < 0, \end{cases} \tag{17}$$

where $d_j \in \mathbb{R}_{++}$ is a constant. Let $\boldsymbol{K}^+(\boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\rm d}}$ denote the tangential stiffness matrix of the structure obtained by neglecting the cable members belonging to $\mathcal{J}$. Note that the slack cables at $\boldsymbol{\xi}^0$ do not contribute to $\boldsymbol{K}^+$, while the contributions of tense cables are included in $\boldsymbol{K}^+$. From (17) and the definition of $\boldsymbol{K}^+$, we see that v defined in (1) is reduced to

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\rm T}\boldsymbol{K}^+(\boldsymbol{\xi}^0)\boldsymbol{u} + \sum_{j \in \mathcal{J}} k_j^{\rm c}(c_j)c_j(\boldsymbol{u})^2. \tag{18}$$

The stability determination problem is formulated as

$$v^* = \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) : \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{19}$$

where the subsidiary conditions (16) and (17) should be satisfied. The following proposition prepares a reformulation of the problem (19) into the form of (3):

Proposition 5.1 ([10, Prop. 3.1]). *Let $\overline{z}_j$ denote the optimal solution of the following problem:*

$$w_j^{\mathrm{c}}(c_j) := \min_{z_j} \left\{ d_j z_j^2 : z_j \geq c_j \right\}. \tag{20}$$

Then $w_j^{\mathrm{c}}(c_j) = k_j^{\mathrm{c}}(c_j) c_j^2$ holds, and $\overline{z}_j$ satisfies

$$\overline{z}_j = \begin{cases} c_j & \text{if } c_j \geq 0, \\ 0 & \text{if } c_j < 0. \end{cases} \tag{21}$$

Define the vector $\boldsymbol{z}_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$ by $\boldsymbol{z}_{\mathcal{J}} = (z_j \mid j \in \mathcal{J})$, which is the sub-vector of $\boldsymbol{z}$ composed of z_j indexed by the set $\mathcal{J}$. Furthermore, let $\boldsymbol{B}_{\mathcal{J}} \in \mathbb{R}^{n^{\mathrm{d}} \times |\mathcal{J}|}$ denote the sub-matrix of $\boldsymbol{B}$ composed of the rows indexed by $\mathcal{J}$. Similarly, let $\boldsymbol{d}_{\mathcal{J}} = (d_j \mid j \in \mathcal{J})$ and $\boldsymbol{D}_{\mathcal{J}} = \mathrm{Diag}(\boldsymbol{d}_{\mathcal{J}})$. By using Proposition 5.1 we can reduce the problem (19) into the form of (3) as follows.

Proposition 5.2 ([10, Prop. 3.2]). *The problem* (19) *is equivalent to the following problem:*

$$\min_{\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}^{+} \boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{D}_{\mathcal{J}} \boldsymbol{z}_{\mathcal{J}} : \boldsymbol{z}_{\mathcal{J}} \geq \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u}, \ \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{22}$$

in the sense that $\overline{\boldsymbol{u}}$ is optimal for (19) *if and only if $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}_{\mathcal{J}})$ satisfying* (21) *with $\boldsymbol{c}_{\mathcal{J}} = \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \overline{\boldsymbol{u}}$ is optimal for* (22)*. Furthermore, The optimal value of* (22) *coincides with v^* defined in* (19)*.*

5.2. Feasibility of problem (22)

According to (5), define a positive definite matrix $\widetilde{\boldsymbol{K}}$ by

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K}^{+} + \widetilde{\lambda} \boldsymbol{I}, \tag{23}$$

where $\widetilde{\lambda}$ is a sufficiently large constant. The perturbed problem (8) defining $\widetilde{v}$ is explicitly obtained as

$$\widetilde{v} = \min_{\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{K}} \boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{D}_{\mathcal{J}} \boldsymbol{z}_{\mathcal{J}} : \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u} \geq \boldsymbol{0}, \ \|\boldsymbol{u}\|^2 - 1 \geq 0 \right\}. \tag{24}$$

Proposition 4.4 verifies to solve (24), instead of (22), in order to determine the stability. The explicit form of (9), which defines g^*, is given by

$$g^*(\lambda) = \max_{\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}} \left\{ \|\boldsymbol{u}\|^2 - 1 : \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}} \boldsymbol{u} \geq \boldsymbol{0}, \ \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{K}} \boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{D}_{\mathcal{J}} \boldsymbol{z}_{\mathcal{J}} \leq \lambda \right\}. \tag{25}$$

According to Theorem 4.7 we solve (25) with $\lambda := \widetilde{\lambda}$ by using Algorithm 4.9 in order to determine stability of the given equilibrium point $\boldsymbol{\xi}^0$.

6. Frictionless unilateral contact problems

Let $\dim \in \{2, 3\}$, and consider a finite-element discretization of an elastic structure in $\mathbb{R}^{\dim}$, which possibly contact with fixed rigid obstacles without friction. The configuration of the structure is described by $\boldsymbol{\xi} \in \mathbb{R}^{n^{\mathrm{d}}}$, which is the position vector of the nodes with respect to the global coordinate system. Some nodes are supposed to be subjected to the unilateral contact constraints, and we denote by $\mathcal{P}_{\mathrm{C}}$ the set of indices of contact candidate nodes.

Let $\boldsymbol{x}^p \in \mathbb{R}^{\dim}$ denote the position vector of the pth node with respect to an appropriately defined reference frame. For each $p \in \mathcal{P}_{\mathrm{C}}$, the surface of the corresponding obstacle is identified by $\{\boldsymbol{x} \in \mathbb{R}^{\dim} \mid \phi^p(\boldsymbol{x}) = 0\}$. The admissible region of the position vector is written as

$$\{\boldsymbol{\xi} \in \mathbb{R}^{n^{\mathrm{d}}} \mid \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) \leq 0 \ (p \in \mathcal{P}_{\mathrm{C}})\}. \tag{26}$$

On each point of the surface, define $\boldsymbol{n}^p(\boldsymbol{x}) \in \mathbb{R}^{\dim}$ by $\boldsymbol{n}^p(\boldsymbol{x}) = \nabla \phi^p(\boldsymbol{x}) / \|\nabla \phi^p(\boldsymbol{x})\|$, which is the unit inner normal vector of the surface. The reaction at the pth node, $r_{\rm n}^p \in \mathbb{R}$, is restricted to be in the direction opposite to $\boldsymbol{n}^p$. The unilateral contact condition is written as

$$\phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) \leq 0, \quad r_{\rm n}^p \leq 0, \quad \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) r_{\rm n}^p = 0, \tag{27}$$

for each $p \in \mathcal{P}_{\rm C}$. Define a partition $\mathcal{P}_{\rm f}$, $\mathcal{P}_0$, and $\mathcal{P}_{\rm r}$ of the set $\mathcal{P}_{\rm C}$ as

$$\begin{aligned}
&\mathcal{P}_{\rm f}(\boldsymbol{\xi}) = \{p \in \mathcal{P}_{\rm C} \mid \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) < 0\}, && \text{[currently not in contact (free)]},\\
&\mathcal{P}_0(\boldsymbol{\xi}) = \{p \in \mathcal{P}_{\rm C} \mid \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) = 0,\ r_{\rm n}^p = 0\}, && \text{[currently in contact without reaction]},\\
&\mathcal{P}_{\rm r}(\boldsymbol{\xi}) = \{p \in \mathcal{P}_{\rm C} \mid \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) = 0,\ r_{\rm n}^p < 0\}, && \text{[currently in contact with reaction]}.
\end{aligned}$$

Let $\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}$ denote the infinitesimal incremental displacement vector defined with respect to the global coordinate system. For each $p \in \mathcal{P}_{\rm C}$ we denote by $u_{\rm n}^p$ the projection of the incremental nodal displacement of the pth node onto the direction of $\boldsymbol{n}^p$. Define $\boldsymbol{g}_{\rm n}^p : \mathbb{R}^{n^{\rm d}} \to \mathbb{R}^{n^{\rm d}}$ by

$$\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}) = \left[\frac{\partial \boldsymbol{x}^p}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi})\right]^{\rm T} \boldsymbol{n}^p(\boldsymbol{x}^p(\boldsymbol{\xi})), \quad \frac{\partial \boldsymbol{x}^p}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}) = \left(\frac{\partial \boldsymbol{x}^p}{\partial \xi_j}(\boldsymbol{\xi}) \mid j = 1, \dots, n^{\rm d}\right).$$

Then the relation between $u_{\rm n}^p$ and $\boldsymbol{u}$ is written as (see, e.g. [14], for more details)

$$u_{\rm n}^p = \boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi})^{\rm T} \boldsymbol{u}. \tag{28}$$

Suppose that the equilibrium point is given as $\boldsymbol{\xi} = \boldsymbol{\xi}^0$. Define the matrices $\boldsymbol{T}_0$ and $\boldsymbol{T}_{\rm r}$ by

$$\boldsymbol{T}_0^{\rm T} = \left(\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}^0) \mid p \in \mathcal{P}_0(\boldsymbol{\xi}^0)\right), \quad \boldsymbol{T}_{\rm r}^{\rm T} = \left(\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}^0) \mid p \in \mathcal{P}_{\rm r}(\boldsymbol{\xi}^0)\right).$$

From (26), (27), and (28) it follows that the admissible set of $\boldsymbol{u}$ is written as

$$\mathcal{A}(\boldsymbol{\xi}^0) = \{\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}} \mid \boldsymbol{T}_0 \boldsymbol{u} \leq \boldsymbol{0},\ \boldsymbol{T}_{\rm r} \boldsymbol{u} = \boldsymbol{0}\}. \tag{29}$$

At the given equilibrium configuration $\boldsymbol{\xi}^0$, let $\boldsymbol{K} \in \mathcal{S}^{n^{\rm d}}$ denote the tangential stiffness matrix, which depends on the curvature of the obstacle surface [11]. Because of the geometrical nonlinearity, $\boldsymbol{K}$ is indefinite in general. It follows from (29) that the stability of $\boldsymbol{\xi}^0$ is determined by solving the following problem:

$$v^* = \min_{\boldsymbol{u}} \left\{\boldsymbol{u}^{\rm T} \boldsymbol{K} \boldsymbol{u} : \boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0),\ \|\boldsymbol{u}\|^2 = 1\right\}. \tag{30}$$

Thus we can reduce the stability determination problem for frictionless contacts in the form of (2). The SOCP problem that is to be solved in our algorithm can be obtained in a manner similar to section 4.3.

7. Directional stability of elastic-plastic trusses

Consider an elastic-plastic truss. The large deformation is considered in general. The tangential stiffness is defined as the sum of the linear and the geometrical stiffness matrices.

At the given equilibrium point $\boldsymbol{\xi}^0$, the yield function of the jth member is dented by $\phi_j(\cdot; \boldsymbol{\xi}^0) : \mathbb{R} \to \mathbb{R}$, where we assume an associated flow rule for simplicity. Note that $\boldsymbol{\xi}^0$ consists of the current nodal coordinates and yield stresses that are path-dependent. The axial force, q_j, of each member should satisfy $\phi_j(q_j; \boldsymbol{\xi}^0) \leq 0$ at $\boldsymbol{\xi}^0$. The compatibility relation between the incremental member elongation c_j and the incremental displacements $\boldsymbol{u}$ is given by (16). Define ν_j by

$$\nu_j = {\rm d}\phi_j(q_j; \boldsymbol{\xi}^0) / {\rm d}q_j.$$

At the yielding state, i.e. $\phi_j(q_j; \boldsymbol{\xi}^0) = 0$, loading and unloading are characterized by $\nu_j c_j > 0$ and $\nu_j c_j < 0$, respectively.

7.1. Directional stability of elastic-plastic trusses

Let q_j^0 denote the axial force at $\boldsymbol{\xi}^0$. Define the partition $\mathcal{J}$ and $\overline{\mathcal{J}}$ of the set of member indices by

$$\mathcal{J} = \left\{j \in \{1, \dots, n^{\rm m}\} \mid \phi_j(q_j^0; \boldsymbol{\xi}^0) = 0\right\}, \quad \overline{\mathcal{J}} = \left\{j \in \{1, \dots, n^{\rm m}\} \mid \phi_j(q_j^0; \boldsymbol{\xi}^0) < 0\right\}.$$

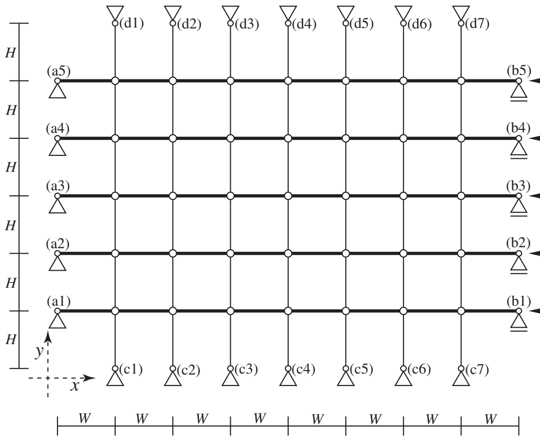

Figure 1: A cable-strut system.

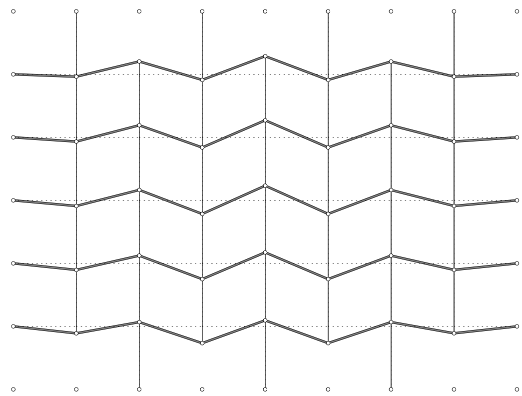

Figure 2: Optimum solution of (24) for the cable-strut system.

Let $k_j(c_j)$ denote the tangential elongation stiffness at $\boldsymbol{\xi}^0$, which is written in the form of

$$k_j(c_j; q_j^0) = \begin{cases} d_j^{\mathrm{p}}, & \text{if } \nu_j c_j \geq 0, \\ d_j^{\mathrm{e}}, & \text{if } \nu_j c_j < 0, \end{cases} \tag{31}$$

where d_j^{e}, $d_j^{\mathrm{p}} \in \mathbb{R}_{++}$ are the constants.

Consider the truss obtained by neglecting the linear stiffnesses of members belonging to $\mathcal{J}$. The tangential stiffness matrix of the obtained structure is denoted by $\boldsymbol{K}^{\mathrm{e}}(\boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\mathrm{d}}}$, which is a constant matrix at the given $\boldsymbol{\xi}^0$. From (31) and the definition of $\boldsymbol{K}^{\mathrm{e}}$, v defined by (1) is written as

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}^{\mathrm{e}}(\boldsymbol{\xi}^0) \boldsymbol{u} + \sum_{j \in \mathcal{J}} k_j(c_j; q_j^0) c_j(\boldsymbol{u})^2. \tag{32}$$

Then the directional stability determination problem, (2), is formulated as

$$v^* = \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) : \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{33}$$

where the subsidiary conditions (16) and (31) should be satisfied. In a manner similar to section 5.2, we can show that (33) is reduced to an optimization problem in the form of (9).

8. Numerical experiments

The stability determination problems (8) and (9) are solved for various structures by using Algorithms 4.11 and 4.9, respectively. We reformulate the subproblem (15) as an SOCP problem, and solve it by using SeDuMi Ver. 1.1 [20, 22], which implements the primal-dual interior-point method for the linear programming problem over symmetric cones. Computation has been carried out on Pentium M (1.2 GHz with 1.0 GB memory) with MATLAB Ver. 7.0.1 [28].

In the following examples, the elastic modulus of structures is 200 GPa, and an initial solution $(\boldsymbol{u}^0, \boldsymbol{z}_{\mathcal{J}}^0)$ for Algorithm 4.9 is generated randomly by using MATLAB built-in-function '*rand*' so that $\|(\boldsymbol{u}^0, \boldsymbol{z}_{\mathcal{J}}^0)\|_\infty \leq 0.5$ is satisfied. At Step 2 of Algorithm 4.11, the optimal solution obtained in the previous iteration is used as an initial solution for Algorithm 4.9. The termination tolerance is chosen as $\epsilon = 10^{-3}$ at Step 0 of Algorithm 4.9. At Step 0 of Algorithm 4.11, we choose $\epsilon = 10^{-4}\widetilde{\lambda}$, $\overline{\lambda}^0 = 2\widetilde{\lambda}$, and $\underline{\lambda}^0 = 0$. The parameter ρ introduced in (10) is chosen as $\rho = 0.1$ for all examples.

Consider a plane cable-strut system illustrated in Figure 1, where $W = 1.0$ m, $H = 1.0$ m, $n^{\mathrm{d}} = 70$, and $n^{\mathrm{m}} = 82$. The nodes (a1)–(a5), (c1)–(c7), and (d1)–(d7) are pin-supported. The displacements of the nodes (b1)–(b5) are constrained in the y-direction. The stability determination problems for the cable-strut structures have been investigated in section 5. See [10] for numerical examples of frictionless contacts and elastoplastic trusses.

The members in the x-direction are struts modeled as truss members, while the members in the y-direction are cables that do not transmit compressive forces. The cross-sectional areas of struts and cables are 5×10^{-3} m^2 and 0.32×10^{-3} m^2, respectively. As for the external force, 1.5 MN is applied in the negative direction of the x-axis at nodes (b1)–(b5). Note that Figure 1 illustrates the deformed equilibrium configuration corresponding to the applied load, i.e. the initial unstressed length of each cable member is equal to H. Therefore, the elongation of each cable member vanishes at the given equilibrium point, and hence $|\mathcal{J}| = 42$. Accordingly, the number of possible combinations in the formulation of the tangential stiffness matrix is 2^{42}.

At the equilibrium point, the smallest eigenvalue of the tangential stiffness matrix $\boldsymbol{K}^{+}$ obtained by neglecting all the cable members is $\lambda_1 = -5.772$. Hence, we choose $\widetilde{\lambda} = 6.060$ (i.e. $\widetilde{\lambda} = 1.05|\lambda_1|$) in (23). For the sake of the stability determination, the problem (25) is solved with $\lambda := \widetilde{\lambda}$ by using Algorithm 4.9 to obtain $g^*(\widetilde{\lambda}) = 1.831 \times 10^{-3}$. Hence, from Corollary 4.10 we can conclude that the equilibrium point is unstable. The CPU time required by Algorithm 4.9 is 8.61 seconds, and 48 SOCP problems are solved.

We next solve (24) in order to obtain the incremental displacements corresponding to the minimal incremental total potential energy. By using Algorithm 4.11, we obtain $\widetilde{v} = 6.049$ and $v^* = -1.147 \times 10^{-2}$ defined in (19). This result verifies that the equilibrium point is unstable in association with Definition 4.3. Figure 2 illustrates the optimal solution obtained, where the slackening cable members have been removed. Algorithm 4.11 requires 15 iterations and 12.8 seconds of the CPU time. In total, 71 SOCP problems are solved, and the average CPU time for solving one SOCP problem is 0.18 seconds. Note that 48 SOCP problems are solved in the first iteration of Algorithm 4.11, while 1.64 SOCP problems in average are required for the remaining iterative steps. This is because the optimal solution of (9) obtained in the previous iteration is used as an initial solution of the next iteration. Notice again that it is sufficient to solve (25) with $\lambda := \widetilde{\lambda}$ in order to determine the stability of the given equilibrium point.

As an alternative case, we choose a slightly larger cross-sectional area 0.33×10^{-3} m^2 for each cable member. The optimal value of (25) with $\lambda := \widetilde{\lambda}$ is computed by using Algorithm 4.9 as $g^*(\widetilde{\lambda}) = -2.459 \times 10^{-2}$. Provided that the obtained solution is globally optimal, Theorem 4.7 implies that the equilibrium point of this case is stable. The problem (24) is solved by using Algorithm 4.11 to obtain $\widetilde{v} = 6.213$ and $v^* = 1.528 \times 10^{-3}$. Thus, the cable-strut system is stabilized by slightly increasing the cross-sectional areas of cables. Moreover, from these two results, we may conjecture that the global optimal solutions of (24) are successfully found.

9. Conclusions

In this paper, we have proposed a numerical technique for determining the stability of the given equilibrium point of structures subjected to the unilateral constraints. We have shown that the stability of a given equilibrium point of a nonsmooth mechanical system can be determined by solving a maximization problem of a convex quadratic function over a convex homogeneous quadratic inequality and some linear inequalities.

In order to solve the presented stability determination problem, we propose a method based on the so-called DC algorithm for a *DC* (difference of convex functions) *programming*. In our algorithm, we solve a sequence of second-order cone programming problems, which can be solved efficiently by using the primal-dual interior-point method.

It has been shown in the numerical examples of various structures subjected to the unilateral constraints that the algorithm presented can find if the given equilibrium point is directionally stable or not. For nonconvex programming problems, there is no practicable global optimal conditions in general, which makes it difficult to check the global optimality of solutions obtained by the proposed algorithm. However, throughout parametric studies it has been shown that the solutions obtained seem to be globally optimal for our numerical examples.

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