8th World Congress on Structural and Multidisciplinary Optimization
June 1-5, 2009, Lisbon, Portugal

Continuation Approach for Investigation of Non-uniqueness of Optimal Topology for Minimum Compliance

Ryo Watada [1] and Makoto Ohsaki [1] *

[1]Department of Architecture and Architectural Engineering,
Kyoto University, Nishikyo, Kyoto 615-8540, Japan
{is.watada,ohsaki}@archi.kyoto-u.ac.jp

1. Abstract
Uniqueness of solution is investigated for topology optimization problem of a plate subjected to in-plane loading condition. The compliance is minimized under constraint on the structural volume. The SIMP approach as well as a filter function is used for preventing convergence to a solution with intermediate densities or a checkerboard pattern. Local non-uniqueness of the solution is defined as a bifurcation of the solution path with respect to the penalization parameter. The optimal solutions are also globally searched from randomly generated initial solutions. It is shown that the global optimal solution cannot always be obtained by continuation with respect to the penalization parameter, although a good approximate solution is found in the numerical examples.

2. Keywords: topology optimization, continuation method, uniqueness of solution, compliance

3. Introduction
There have been numerous researches on topology optimization of plate (sheet), which is discretized to finite elements, and subjected to in-plane static loads. In most of researches, the compliance is minimized under constraint on the total structural volume. The 0-1 thickness variable is relaxed to a real variable, and the standard nonlinear programming (NLP) is used to find the optimal topology after removing the element with vanishing thickness. However, simple application of NLP results in the solution with intermediate thickness in many elements, which is called gray solution. Therefore, the intermediate thickness is penalized to have small stiffness using, e.g., the solid isotropic material with penalization (SIMP) approach [1], or penalized to have artificially large structural volume [2]. The homogenization method is also effective for preventing gray solutions [3, 4]. Another difficulty of checkerboard solution is avoided using the penalization in perimeter length [5], or in the gradient of the thickness [6]. A smoothing filter of the thickness [2, 7] can be used for regularizing the problem. Optimal topology can be controlled by varying the parameter for the filter function. Rirtz [8] obtained various topologies for different values of gradient parameter.

In the SIMP approach, larger penalization leads to a 0-1 solution; however, the convergence property is deteriorated if the penalization parameter is too large, because the convexity of the objective function is lost. Therefore, the optimal solutions are traced by gradually increasing the penalization parameter based on the so called *continuation method*. In the rigorous definition of continuation method, the path of the solutions is traced based on the differentiation of the governing equations [9], which is basically same as the parametric programming approach [10, 11] or homotopy method [12] for tracing the optimal solutions corresponding to the various parameter values [13]. However, in most of the continuation methods for the plate (sheet) topology optimization problems, the governing equations are not differentiated, and the solutions are found consecutively with increasing value of the penalization parameter. Stolpe and Svanberg [14] investigated the trajectory of the optimal solutions with respect to the penalization parameter for a problem for minimizing the worst value of compliances under multiple loading conditions. It was shown that the integer solutions are local optima that may be discontinuous with respect to the penalization parameter, and the continuation method will not produce the global optimal solution.

Uniqueness and stability of the optimal solution have been studied by many researchers. Jog and Haber [15] derived the conditions of stability using incremental form of the variational problem. Petersson [16] investigated convergence of the solution with respect to the mesh size, and noted that the objective function may be insensitive to the thickness variation for special loading condition with bilinear elements.

*Corresponding author.

In this study, we first define local non-uniqueness of the solution as a bifurcation of the solution path with respect to the penalization parameter. The formulations for numerical continuation with respect to the penalization parameter is rigorously derived by differentiating the Karush-Kuhn-Tucker (KKT) conditions and the stiffness (equilibrium) equations, and the conditions for local uniqueness of the solution is derived as the singularity of the Jacobian of the governing equations [17–19]. Therefore, the proposed procedure is applicable to any types of optimization problems. It is shown that the globally optimal solution cannot be obtained by continuation with respect to the penalization parameter; hence, the simple continuation without differentiation of the governing equations is not valid for this problem. We show that nonuniqueness exists in the displacements, although the thicknesses may be locally unique as discussed by Kočvara and Outrata [20]. Next, the optimal topologies are also globally searched from many randomly assigned initial solutions to show that various local optimal solutions with different topologies and almost same objective function value exist for this problem.

4. Continuation method and bifurcation of solution path

Consider a problem of finding the solution $\boldsymbol{x} \in \mathbb{R}^q$ defined by the nonlinear equation

$$\boldsymbol{F}(\boldsymbol{x}, t) = \boldsymbol{0} \tag{1}$$

where t is a parameter, and $\boldsymbol{F}(\boldsymbol{x}, t) \in \mathbb{R}^q$ is continuously differentiable with respect to $\boldsymbol{x}$ and t. Since the solution is defined by (1) for each specified value of t, it is regarded as a function of t, and is denoted by $\tilde{\boldsymbol{x}}(t)$. Suppose we have a solution $\tilde{\boldsymbol{x}}(t^0)$ for $t = t^0$. Then the solution for $t = t^0 + \Delta t$ can be linearly approximated as

$$\boldsymbol{J}\left[\tilde{\boldsymbol{x}}(t^0 + \Delta t) - \tilde{\boldsymbol{x}}(t^0)\right] + \frac{\partial \boldsymbol{F}}{\partial t}\Delta t = \boldsymbol{0} \tag{2}$$

where $\boldsymbol{J} \in \mathbb{R}^{q \times q}$ is the Jacobian of $\boldsymbol{F}$, for which the (i, j)-component J_{ij} is defined as

$$J_{ij} = \frac{\partial F_i}{\partial x_j} \tag{3}$$

Bifurcation of the solution path exists if $\boldsymbol{J}$ is singular [18, 19]. If $\boldsymbol{J}$ is non-singular, then the solution path can be found using a Newton-Raphson-type iterative correction of the governing equation (1) in a similar manner as the arc-length method for geometrically nonlinear equilibrium analysis.

Jog and Haber [15] derived the conditions of stability using incremental form of the variational problem for a min-max optimization problem. However, our approach for investigation of uniqueness of optimal solution is applicable to any type of nonlinear programming problem.

5. Optimization problem and optimality conditions

Consider a plate discretized to finite elements, and subjected to in-plane loading. The number of elements and the number of degrees of freedom are denoted by m and n, respectively. Let d_j denote the thickness of the jth element, for which the upper and lower bounds are given as

$$d_j^{\mathrm{L}} \leq d_j \leq 1, \quad (j = 1, \dots, m) \tag{4}$$

where d_j^{L} is a sufficiently small positive value. We investigate the effect of a filter function on uniqueness of the solution. As an example, we use the filter by Bruns [2], where function ω_j is defined as

$$\omega_j(s_{ij}) = \begin{cases} \dfrac{1}{(2/3)\pi r} \exp\left(-\dfrac{s_{ij}^2}{2(r/3)^2}\right) & \text{for } s_{ij} \leq r \\ 0 & \text{for } s_{ij} > r \end{cases} \tag{5}$$

where r is the radius of filter, and s_{ij} is the distance between the centers of the ith and jth elements. The thickness ϕ_i of the ith element is defined as the function of $\boldsymbol{d} = (d_1, \dots, d_m)^\top$ as

$$\phi_i(\boldsymbol{d}) = \sum_{j=1}^{m} \Omega_{ij} d_j, \quad \Omega_{ij} = \frac{\omega_j(s_{ij})}{\omega_i}, \quad \omega_i = \sum_{j=1}^{m} \omega_j(s_{ij}) \tag{6}$$

Note that $\Omega_{ij} = 1$ for $i = j$ and 0 for $i \neq j$ corresponds to the formulation without filter.

The objective function to be minimized is the compliance $C(\boldsymbol{d})$ given as

$$C(\boldsymbol{d}) = \boldsymbol{P}^\top \boldsymbol{U}(\boldsymbol{d}) \tag{7}$$

where $\boldsymbol{P} \in \mathbb{R}^n$ is the specified external load vector, and the nodal displacement vector $\boldsymbol{U} \in \mathbb{R}^n$ is obtained from the following equilibrium equation:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{8}$$

where $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ is the stiffness matrix.

We use the SIMP approach to prevent gray optimal solutions. The stiffness matrix $\boldsymbol{K}$ is defined by the matrix $\boldsymbol{K}_i \in \mathbb{R}^{n\times n}$ corresponding to the unit thickness of the ith element and the penalization parameter p as

$$\boldsymbol{K} = \sum_{i=1}^{m} \phi_i^p \boldsymbol{K}_i \tag{9}$$

Then the problem of minimizing the compliance is formulated with respect to the variable vector $\boldsymbol{d}$ as

$$\text{minimize} \quad C(\boldsymbol{d}) = \boldsymbol{P}^\top \boldsymbol{U}(\boldsymbol{d}) \tag{10a}$$

$$\text{subject to} \quad \sum_{i=1}^{m} \phi_i(\boldsymbol{d}) = \bar{V} \tag{10b}$$

$$d_j^{\mathrm{L}} \leq d_j \leq 1, \quad (j = 1, \dots, m) \tag{10c}$$

where $\bar{V}$ is the specified value of the total structural volume.

The Lagrangian of problem (10) is formulated as

$$L(\boldsymbol{d}, \lambda, \boldsymbol{\mu}^{\mathrm{U}}, \boldsymbol{\mu}^{\mathrm{L}}) = C(\boldsymbol{d}) + \lambda \left(\sum_{i=1}^{m} \phi_i(\boldsymbol{d}) - \bar{V} \right) + \sum_{j=1}^{m} \mu_j^{\mathrm{U}} (d_j - 1) + \sum_{j=1}^{m} \mu_j^{\mathrm{L}} (-d_j + d_j^{\mathrm{L}}) \tag{11}$$

where λ, μ_j^{U} and μ_j^{L} are the Lagrange multipliers. By differentiating (11) with respect to d_j, we have

$$\frac{\partial L}{\partial d_j} = -\sum_{i=1}^{m} \Omega_{ij} p \phi_i^{p-1}(\boldsymbol{d}) \boldsymbol{U}^\top \boldsymbol{K}_i \boldsymbol{U} + \lambda \sum_{i=1}^{m} \Omega_{ij} + \mu_j^{\mathrm{U}} - \mu_j^{\mathrm{L}} \tag{12}$$

Define $G(\boldsymbol{d})$ as

$$G(\boldsymbol{d}) = \sum_{i=1}^{m} \Omega_{ij} p \phi_i^{p-1}(\boldsymbol{d}) \boldsymbol{U}^\top \boldsymbol{K}_i \boldsymbol{U} - \lambda \sum_{i=1}^{m} \Omega_{ij} \tag{13}$$

Then the optimality conditions (KKT conditions) are derived as

$$G(\boldsymbol{d}) \begin{cases} = 0 & \text{for} \quad d_j^{\mathrm{L}} < d_j < 1 \\ \geq 0 & \text{for} \quad d_j = 1 \\ \leq 0 & \text{for} \quad d_j = d_j^{\mathrm{L}} \end{cases} \tag{14}$$

The set of s elements satisfying $G(\boldsymbol{d}) = 0$ with equality is denoted by $\mathcal{I}$; i.e.,

$$\mathcal{I} = \left\{ i \mid d_i^{\mathrm{L}} < d_i < 1 \right\} \tag{15}$$

6. Sensitivity of optimal solution with respect to penalization parameter

Equations for sensitivity analysis of the optimal solutions with respect to p are derived below, where $(\cdot)'$ indicates differentiation with respect to p. We use the following relation:

$$(\phi_i^p)' = \phi_i^p \ln \phi_i + \phi_i^{p-1} p \sum_{j=1}^{m} \Omega_{ij} d_j' \tag{16}$$

By differentiating (8) with respect to p, we have

$$\boldsymbol{K}\boldsymbol{U}' + \sum_{i=1}^{m} \sum_{j=1}^{m} p \phi_i^{p-1} \Omega_{ij} d_j' \boldsymbol{K}_i \boldsymbol{U} = -\sum_{i=1}^{m} \phi_i^p \ln \phi_i \boldsymbol{K}_i \boldsymbol{U} \tag{17}$$

By differentiating the volume constraint (10b) and multiplying $-1/2$, we obtain

$$-\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \Omega_{ij} d_j' = 0 \tag{18}$$

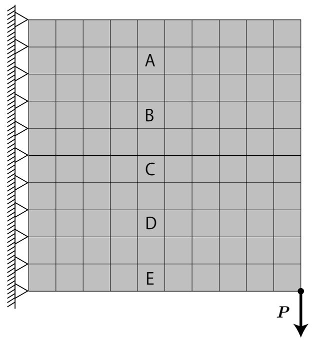


Figure 1: A 10×10 plate model.

Suppose the active constraints remain active in the neighborhood of the current parameter value. Then, for the elements $d_l^{\mathrm{L}} < d_l < 1$ $(l \in \mathcal{I})$, differentiation of $G(\boldsymbol{d}) = 0$ leads to

$$\begin{aligned}\sum_{i=1}^{m} p\phi_i^{p-1}\Omega_{il}\boldsymbol{U}^\top \boldsymbol{K}_i \boldsymbol{U}' + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} p(p-1)\phi_i^{p-2}\Omega_{il}\Omega_{ij}d_j'\boldsymbol{U}^\top\boldsymbol{K}_i\boldsymbol{U} - \frac{1}{2}\lambda'\sum_{i=1}^{m}\Omega_{il} \\ = -\frac{1}{2}\sum_{i=1}^{m} p\phi_i^{p-1}\ln\phi_i\Omega_{il}\boldsymbol{U}^\top\boldsymbol{K}_i\boldsymbol{U} - \frac{1}{2}\sum_{i=1}^{m}\phi_i^{p-1}\Omega_{il}\boldsymbol{U}^\top\boldsymbol{K}_i\boldsymbol{U}, \quad (l \in \mathcal{I})\end{aligned} \tag{19}$$

For $i \notin \mathcal{I}$, we have $d_i' = 0$. Therefore, there are $n+s+1$ linear equations (17), (18) and (19) for $n+s+1$ variables $\boldsymbol{U}'$, d_i' $(i \in \mathcal{I})$ and λ'.

The member numbers are rearranged so that $\mathcal{I} = \{1, \dots, s\}$, and define $\boldsymbol{d}_0 = (d_1, \dots, d_s)^\top$. Then the linear equations for computing the parametric sensitivity coefficients of the optimal solution are written as

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{B}^{12} & \boldsymbol{0} \\ \boldsymbol{B}^{12\top} & \boldsymbol{B}^{22} & \boldsymbol{B}^{23} \\ \boldsymbol{0}^\top & \boldsymbol{B}^{23\top} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{U}' \\ \boldsymbol{d}_0' \\ \lambda' \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}^1 \\ \boldsymbol{b}^2 \\ 0 \end{pmatrix} \tag{20}$$

which is simply written as

$$\boldsymbol{B}\boldsymbol{X}' = \boldsymbol{b} \tag{21}$$

Let $H_i = \boldsymbol{U}^\top\boldsymbol{K}_i\boldsymbol{U}$, and $\boldsymbol{b}^2 = (b_1^2, \dots, b_s^2)^\top$. The (i,j)-components of $\boldsymbol{B}^{12} \in \mathbb{R}^{n\times s}$, $\boldsymbol{B}^{22} \in \mathbb{R}^{s\times s}$, and ith component of $\boldsymbol{B}^{23} \in \mathbb{R}^s$ are denoted by B_{ij}^{12}, B_{ij}^{22}, and B_i^{23}, respectively. The jth column of $\boldsymbol{K}_i$ is denoted by $\boldsymbol{k}_{ij} \in \mathbb{R}^n$. Then the components of the symmetric matrix $\boldsymbol{B} \in \mathbb{R}^{(n+s+1)\times(n+s+1)}$ and the constant vector $\boldsymbol{b} \in \mathbb{R}^{n+s+1}$ are given as

$$B_{ij}^{12} = p\sum_{k=1}^{m}\phi_k^{p-1}\Omega_{kj}\boldsymbol{k}_{ki}^\top\boldsymbol{U}, \quad B_{ij}^{22} = \frac{1}{2}p(p-1)\sum_{k=1}^{m}\phi_k^{p-2}\Omega_{ki}\Omega_{kj}H_k, \quad B_i^{23} = -\frac{1}{2}\sum_{k=1}^{m}\Omega_{ki} \tag{22}$$

$$\boldsymbol{b}^1 = -\sum_{i=1}^{m}\phi_i^p\ln\phi_i\boldsymbol{K}_i\boldsymbol{U}, \quad b_i^2 = -\frac{1}{2}\sum_{k=1}^{m}\left\{\Omega_{ki}(1+p\ln\phi_k)\right\}\phi_k^{p-1}H_k \tag{23}$$

The optimal solutions can be traced successively solving (21), where the set of elements in $\mathcal{I}$ should be appropriately updated [21]. Since $\boldsymbol{B}$ is symmetric, the stability of solution is detected from the eigenvalues or the condition number of the matrix.

We can solve $\boldsymbol{U}'$ using the first equation of (20) as

$$\boldsymbol{U}' = \boldsymbol{K}^{-1}\boldsymbol{b}^1 - \boldsymbol{K}^{-1}\boldsymbol{B}^{12}\boldsymbol{d}_0' \tag{24}$$

which is incorporated to the second and third equations of (20) to obtain

$$\begin{pmatrix} \boldsymbol{B}^{22*} & \boldsymbol{B}^{23} \\ \boldsymbol{B}^{23\top} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{d}' \\ \lambda' \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}^2 - \boldsymbol{B}^{12\top}\boldsymbol{K}^{-1}\boldsymbol{b}^1 \\ 0 \end{pmatrix}, \quad \boldsymbol{B}^{22*} = \boldsymbol{B}^{22} - \boldsymbol{B}^{12\top}\boldsymbol{K}^{-1}\boldsymbol{B}^{12} \tag{25}$$

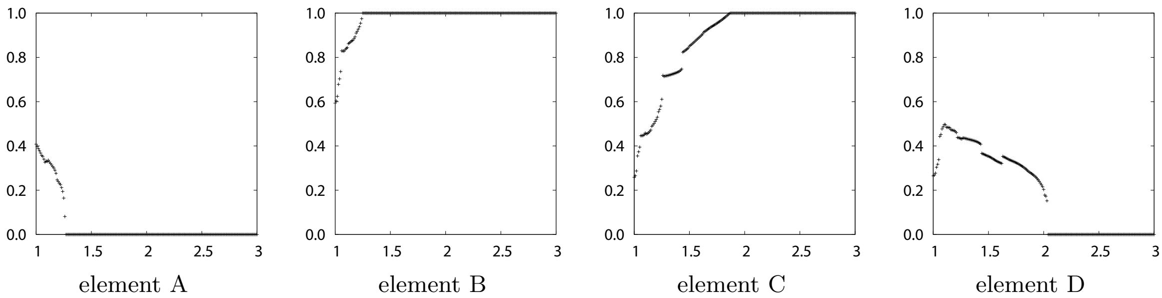


Figure 2: Solution path of d_j of elements A, B, C and D with respect to p (without filter).

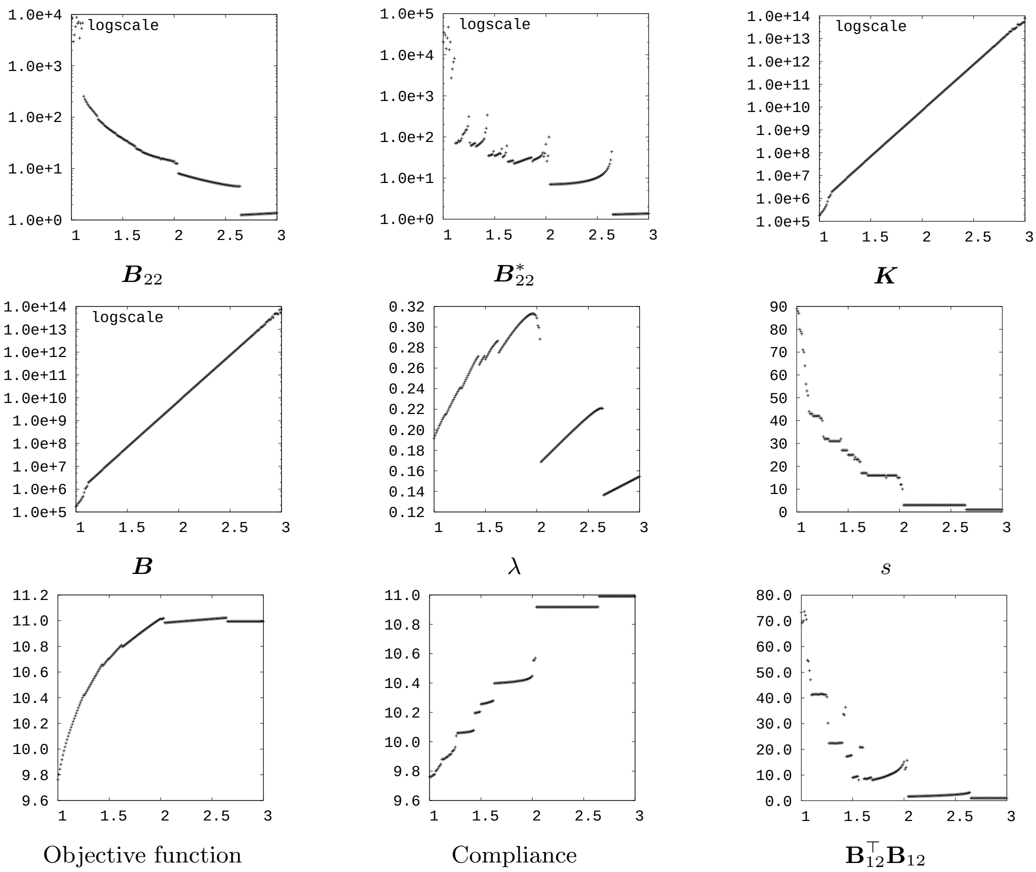


Figure 3: Variations of condition numbers and objective function (without filter).

Note that the vector $\boldsymbol{B}^{23}$ is nonzero, and all components are 1/2 if filter is not used. Therefore, it can be easily shown that the solution is unique if $\boldsymbol{B}^{22*}$ is non-singular.

Jog and Haber [15] derived the conditions of uniqueness and stability of the optimal solution such that $\boldsymbol{K}$ and $\boldsymbol{B}^{22*}$ are positive definite based on incremental form of the variational form of the min-max optimization problem. Note that the sign of $\boldsymbol{B}^{22*}$ in [15] is different from $\boldsymbol{B}^{22*}$ in (20), because the definition of the Lagrangian is different. It should also be noted that the conditions for uniqueness of the solution has been derived using a simple manner based on the stability of the solution path. Furthermore, the singularity of $\boldsymbol{K}$ leads only to non-uniqueness of the displacements that may not have any effect of non-uniqueness of the thickness variables [20].

7. Numerical examples

Consider a square plate as shown in Fig. 1 subjected to the concentrated load $P = 10$. The total size of the plate is 1000×1000, and is discretized to 100 square elements with bilinear displacement fields and

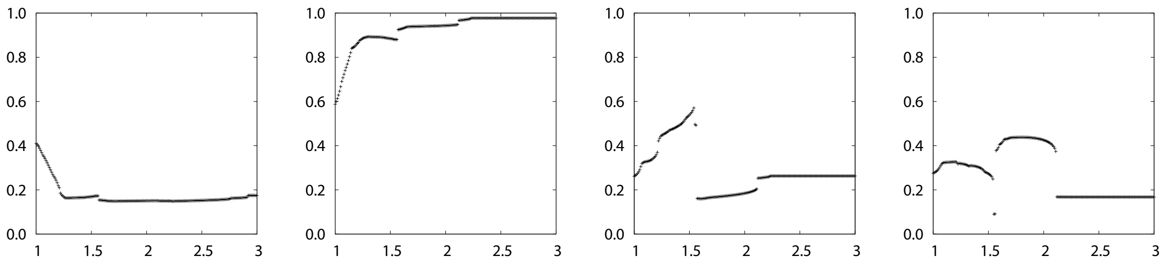


Figure 4: Solution path of ϕ_j of elements A, B, C and D with respect to p (with filter).

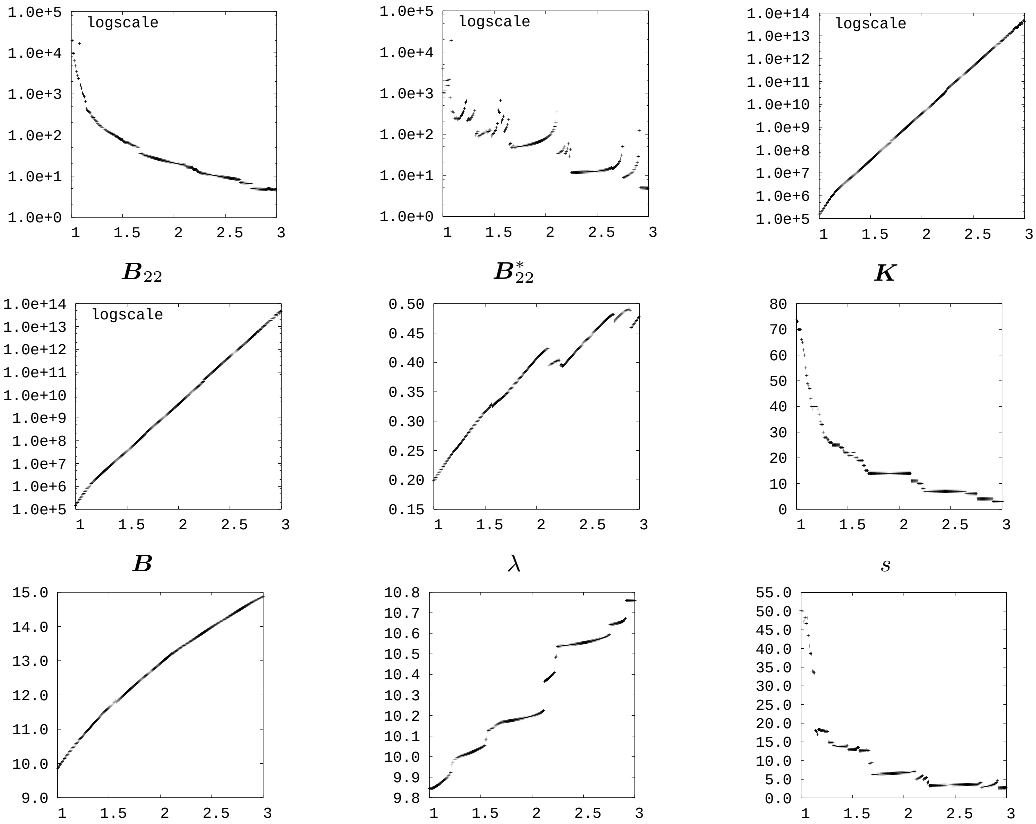
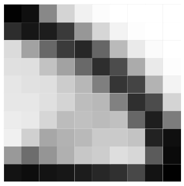


Figure 5: Variations of condition numbers and objective function (with filter).

constant thickness. The elastic modulus is 200, and the specified structural volume is 4000. The lower bound of the thickness is 0.0001. Optimization is carried out by SNOPT Ver. 7 [22], where the sequential quadratic programming (SQP) is used. The default values are used for the parameters except the strict tolerance 10^{-12} for feasibility and optimality of the solution.

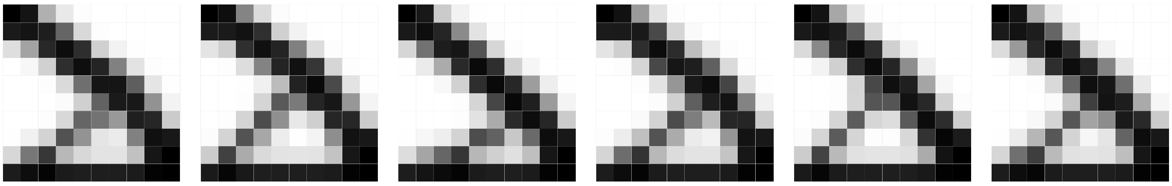
The formulation of sensitivity of optimal solution with respect to the parameter p is first confirmed for the case with filter function, where the radius r of the filter is 150. For $p = 1.0$, the optimal thickness of the elements A, B, C, D and E indicated in Fig. 1 are $d_{\mathrm{A}} = 0.360639$, $d_{\mathrm{B}} = 0.574237$, $d_{\mathrm{C}} = 0.291846$, $d_{\mathrm{D}} = 0.318318$ and $d_{\mathrm{E}} = 1.000000$. The sensitivity coefficients are obtained as $d'_{\mathrm{A}} = -0.2614$, $d'_{\mathrm{B}} = 1.8565$, $d'_{\mathrm{C}} = 0.8852$, $d'_{\mathrm{D}} = 0.5600$ and $d'_{\mathrm{E}} = 0$. The sensitivity coefficients obtained by the forward finite difference with $\Delta p = 0.001$ are $d'_{\mathrm{A}} = -0.266$, $d'_{\mathrm{B}} = 1.86$, $d'_{\mathrm{C}} = 0.894$, $d'_{\mathrm{D}} = 0.565$ and $d'_{\mathrm{E}} = 0$, which agree in good accuracy with the analytical results.

Consider the case without filter. Optimal solutions are found for the parameters between $p = 1.0$ and 3.0 with the increment $\Delta p = 0.01$, by tracing the solution path assigning the solution of $p - \Delta p$ as the



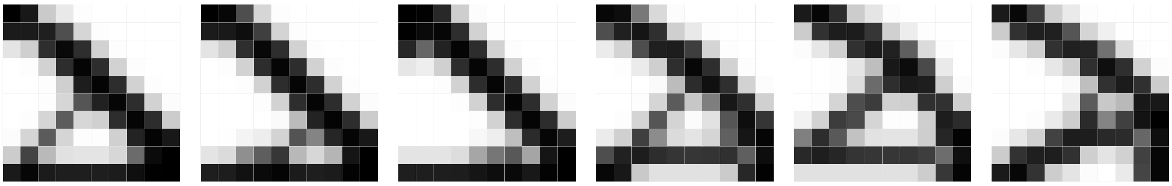
C=9.845

Figure 6: Global optimal solution for $p = 1.0$.



C=12.92 C=13.02 C=12.95 C=12.93 C=13.04 C=12.93

Figure 7: Local optimal solutions for $p = 2.0$.



C=15.05 C=14.83 C=14.77 C=18.39 C=20.10 C=18.95

Figure 8: Typical local optimal solutions for $p = 3.0$.

initial solution for the SQP algorithm. Fig. 2 shows the path of the variables d_j of elements A, B, C and D. As is seen, the optimal solutions are piecewise continuous functions of p, where discontinuity exists at the parameter value when the number s of the elements in $\mathcal{I}$ changes.

The condition numbers in log scale, the number s of variables in $\mathcal{I}$, the objective function and compliance are plotted in Fig. 3. For $p = 2.0$, the condition number of $\boldsymbol{B}$ is 7.10187×10^9, $\lambda = 0.3087$ and $s = 15$. Note that the condition number of $\boldsymbol{K}$ is 7.10183×10^9, which is very large. Therefore, the singularity of $\boldsymbol{B}$ is due to that of $\boldsymbol{K}$; i.e., non-uniqueness exists in $\boldsymbol{U}$. By contrast, the condition numbers of $\boldsymbol{B}_{22}$ and $\boldsymbol{B}^*_{22}$ are 13.973 and 66.771, respectively, which are sufficiently small. Therefore, the solution is locally unique. It is also seen from Fig. 3 that the matrix $\boldsymbol{B}^{12}$ is full (column) rank with $s < n$.

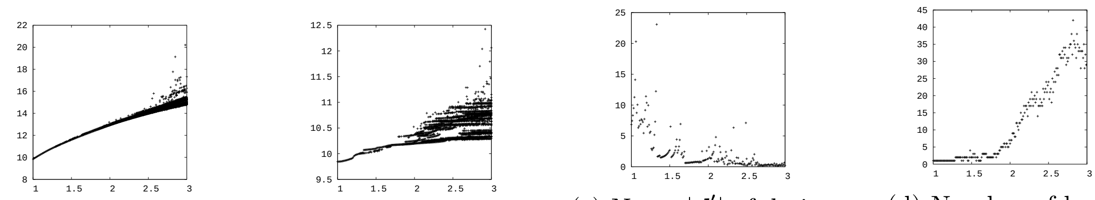
For $p = 1.1$ without filter, $\lambda = 0.2135$, $s = 53$, and the condition numbers for $\boldsymbol{K}$, $\boldsymbol{B}_{22}$, $\boldsymbol{B}^*_{22}$ and $\boldsymbol{B}$ are 1.22391×10^6, 5340.2, 6672.9 and 1.16153×10^6, respectively. Therefore, no bifurcation is found for the solution path of the optimal thickness variables at $p = 1.1$.

It should be noted that the condition number of $\boldsymbol{B}^*_{22}$ is very small, although $\boldsymbol{K}$ is ill-conditioned. This fact corresponds to the condition for uniqueness of the solution such that $\boldsymbol{K}$ is positive definite in Ker $\boldsymbol{B}^{12\top}$, together with the fact that $\boldsymbol{B}^{12}$ is full rank.

We next consider the case with filter function, where the radius r of the filter is 150. Fig. 4 shows the path of the variables d_j of elements A, B, C and D, which are also piecewise continuous functions of p. Note that the possibility of obtaining gray solutions increases with the use of filter function. The condition numbers, the number s of variables in $\mathcal{I}$, the objective function and compliance, which is computed from the thickness without penalization, are plotted in Fig. 5.

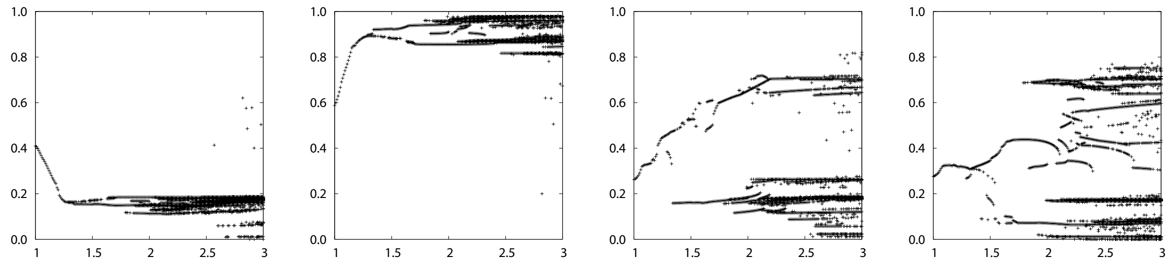
8. Global search of optimal solutions

We generate various local optimal solutions from 100 random initial solutions using filter functions with radius 150. The number of local optimal solutions is 6 for $p = 2.0$, and 31 for $p = 3.0$. The two solutions are considered to be different if the difference in the thickness is more than 0.1 in at least one element. The



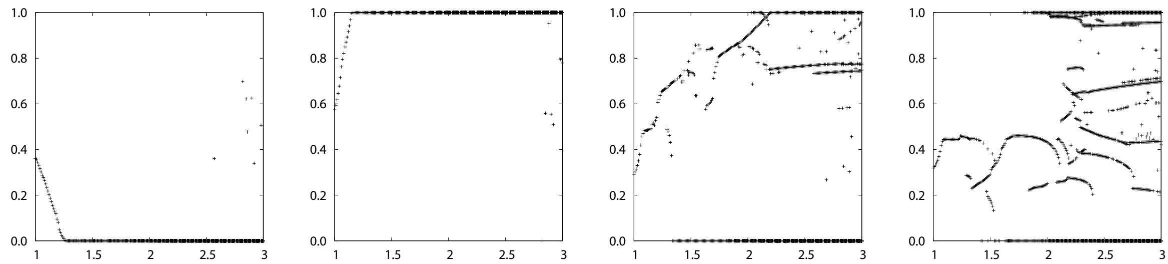
(a) Objective value (b) Compliance (c) Norm $|\boldsymbol{d}'|$ of design variable (d) Number of local optimal solutions

Figure 9: Objective value



element A element B element C element D

Figure 10: Solution path of ϕ_j of elements A, B, C and D with respect to p.



element A element B element C element D

Figure 11: Solution path of d_j of elements A, B, C and D with respect to p.

typical solutions for $p = 1.0$, 2.0, and 3.0 are shown in Figs. 6–8, respectively. Note that the values of the objective function and compliance of the best solution at $p = 3.0$ are 14.8815 and 10.7597, respectively, while those in Fig. 5 obtained by continuation are 14.7653 and 10.2937, respectively. Therefore, a good approximate solution has been found in this case, although the best solution could not be found.

Since the objective function is convex for $p = 1.0$, the same global solution has been found from 100 different initial solutions irrespective of the filter radius, and also for the case without filter function. It has also been confirmed that best solution from randomly generated initial solutions for $p = 2.02$ can be obtained from the best solution of $p = 2.00$ as the initial solution for SQP. The similar results have been confirmed for the solutions for $p = 2.52$ and 3.00 obtained from the initial solutions corresponding to $p = 2.50$ and 2.98, respectively; therefore, continuation approach is effective for this problem, if a small value is chosen for the increment of p.

The variation of the objective function, compliance, norm $|\boldsymbol{d}'|$ of sensitivity coefficient of the optimal solution, and the number of local optimal solutions are plotted in Fig. 9. Note that the norm is plotted only for the solution that has the smallest objective value at each parameter value. As is seen, the objective function and compliance of the local optimal solutions are widely distributed if p is large. However, no divergence has been found for the solution path, because $|\boldsymbol{d}'|$ has finite values. Therefore, non-uniqueness of the solution is due to the existence of many isolated local optimal solutions. The optimal thickness and design variable for various values of p between 1.0 and 3.0 are shown in Figs. 10 and 11, respectively.

9. Conclusion

A simple formulation has been presented for investigating local non-uniqueness of the optimal solution of a plate that minimizes the compliance under specified in-plane load and the total structural volume. The plate is discretized to finite elements with bilinear displacement interpolation and constant thickness. The conditions of non-uniqueness of the solution are derived based on a bifurcation of the solution path with respect to the penalization parameter of the SIMP approach. The formulations for numerical continuation with respect to the penalization parameter is rigorously derived by differentiating the KKT conditions and the stiffness (equilibrium) equations.

It has been shown that the globally optimal solution cannot be obtained by continuation with respect to the penalization parameter, although a good approximate solution has been found in the particular numerical example. The condition numbers of the matrix for computing the sensitivity coefficients of the optimal solutions, and the corresponding sub-matrices, have been computed to show that non-uniqueness of the nodal displacements may seriously deteriorates the accuracy of the solution path, although no spurious mode has been found in this example.

The optimal topologies have also been globally searched from many randomly assigned initial solutions to show that various local optimal solutions with different topologies and almost same objective function value exist for this problem.

11. References

[1] M. P. Bendsøe. Optimal shape design as a material distribution problem. *Structural Optimization*, 1:193–202, 1989.

[2] T. E. Bruns. A reevaluation of the SIMP method with filtering and an alternative formulation for solid–void topology optimization. *Struct. Multidisc. Optim.*, 30:428–436, 2005.

[3] M. P. Bendsøe and N. Kikuchi. Generating optimal topologies in structural design using a homogenization method. *Comp. Meth. Appl. Mech. Engng.*, 71:197–224, 1988.

[4] K. Suzuki and N. Kikuchi. A homogenization method for shape and topology optimization. *Comp. Meth. Appl. Mech. Engng.*, 93:291–318, 1991.

[5] J. Petersson. Some convergence results of perimeter-controlled topology optimization. *Comp. Meth. Appl. Mech. Engng.*, 171:121–140, 1999.

[6] J. Ptersson and O. Sigmund. Slope constrained topology optimization. *Int. J. Num. Meth. Engng.*, 41:1417–1434, 1998.

[7] T. E. Bruns. Topology optimization by penalty (TOP) method. *Comp. Meth. Appl. Mech. Engng.*, 196:4430–4443, 2007.

[8] A. Rietz. Modified conditions for minimum of compliance. *Comp. Meth. Appl. Mech. Engng.*, 196:4413–4418, 2007.

[9] H. D. Mittelmann and D. Roose, editors. *Continuation Techniques and Bifurcation Problems.* Birkhäuser Verlag, Basel, Germany, 1990.

[10] T. Gal. *Postoptimal Analyses, Parametric Programming, and Related Topics.* Mc-Graw Hill, 1979.

[11] A. V. Fiacco. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming.* Academic press, 1983.

[12] L. T. Watson and R. T. Haftka. Modern homotopy methods in optimization. *Comp. Meth. Appl. Mech. Engng.*, 74(3):289–305, 1989.

[13] T. Nakamura and M. Ohsaki. Sequential optimal truss generator for frequency ranges. *Comp. Meth. Appl. Mech. Engng.*, 67(2):189–209, 1988.

[14] M. Stolpe and K. Svanberg. On the trajectories of the epsilon-relaxation approach for stress-constrained truss topology optimization. *Struct. Multidisc. Optim.*, 21:140–151, 2001.

[15] C. S. Jog and R. B. Haber. Stability of finite element models for distributed-parameter optimization and topology design. *Comp. Meth. Appl. Mech. Engng.*, 130:203–226, 1996.

[16] J. Petersson. A finite element analysis of optimal variable thickness sheets. *SIAM J. Mumer. Anal.*, 36(6):1759–1778, 1999.

[17] M. Ohsaki. Local and global searches of approximate optimal designs of regular frames. *Int. J. Num. Meth. Engng.*, 67:132–147, 2006.

[18] K. Ikeda and K. Murota. *Imperfect Bifurcation Phenomena in Structures and Materials – An Engineering Use of Group-theoretic Bifurcation Theory.* Appl. Math. Sci. Ser. 149. Springer, New York, NY, 2002.

[19] M. Ohsaki and K. Ikeda. *Stability and Optimization of Structures – Generalized Sensitivity Analysis.* Mechanical Engineering Series. Springer, New York, 2007.

[20] M. Kočvara and J. V. Outrata. Effective reformulation of the truss topology design problem. *Optimization and Eng.*, 7:201–219, 2006.

[21] M. Ohsaki and T. Nakamura. Minimum constraint perturbation method for topology optimization of systems. *Eng. Opt.*, 26:171–186, 1996.

[22] P. E. Gill, W. Murray, and M. A. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. *SIAM Journal on Optimization*, 12:979–1006, 2002.