STABILITY OF TENSEGRITY STRUCTURES WITH DIHEDRAL SYMMETRY

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ABSTRACT

This study shows that using symmetry can be of great benefit in an investigation of the stability of tensegrity structures: we are able to find the general super stability condition for a class of structures with the same type of symmetry. In the study, we are focusing on the prismatic as well as star tensegrity structures, both classes of structures are of dihedral symmetry. Making use of their symmetry properties, the force density matrix can be analytically transformed into block-diagonal form, where the sub-matrices in the leading diagonal are independent. Positive semi-definiteness of the force density matrix, which is the necessary condition for super stability, can then be easily verified by considering the sub-matrices instead of the original matrix. This enables us to find the super stability conditions for prismatic and star structures: a prismatic tensegrity structure is super stable if and only if its horizontal cables are connected to adjacent nodes; and a star structure is super stable if and only if it consists of odd numbers of struts, while the struts are as close to each other as possible.

1. INTRODUCTION

In this study, we investigate *super stability* of prismatic and 'star' tensegrity structures, that are of dihedral symmetry. A super stable structure is guaranteed to be stable (having locally isolated minimum of strain energy), for any level of self-stress and any material properties, as long as every member has a positive 'modified' axial stiffness [1]. The simplest examples of prismatic and star structures are shown in Fig. 1. There is a clear link between these two classes of structures: the horizontal cables in each of the two parallel circles containing the nodes in a prismatic structure are replaced by a star of cables in a star structure, with a new centre node. Indeed, we shall see that the equilibrium positions of the nodes, and self-stress forces in the vertical cables and the struts, are identical in both structures.

The prismatic structure has at least one non-rigid-body infinitesimal mechanism, resulting from the existence of the self-stress mode. However, the star structure has many more infinitesimal mechanisms: at each of the boundary nodes, a strut is in equilibrium with two cables, all

of which must therefore lie in a plane; thus, out-of-plane movement of the node must be an infinitesimal mechanism, and there are at least six infinitesimal mechanisms—in fact there is another infinitesimal mechanism corresponding to the self-stress mode. In spite of the existence of infinitesimal mechanisms, we will show that many prismatic and star tensegrity structures are *super stable* if particular conditions are satisfied.

As will be discussed later, super stability of a tensegrity structure is related to the positive semi-definiteness of the force density matrix (sometimes called the 'small' stress matrix, for example in [1]). The super stability condition for prismatic structures has been obtained in [2] making use of the special properties of the force density matrix as a circulant matrix. In this study, we present another systematic way to analytically block-diagonalise the force density matrix into sub-matrices of only one or two dimensions, making use of their dihedral symmetry. Using the same methodology, we further prove that a star structure is super stable as long as it is composed of odd number of struts and the struts are closest to each other.

Following this introduction, the paper is organized as follows: Section 2 uses the symmetry of a structure to find its self-stress forces in the state of self-equilibrium. Section 3 block-diagonalises the force density matrix and finds the condition for super stability of the prismatic and star structures. Section 4 briefly concludes the study.

2. SYMMETRY AND SYMMETRY AND SELF-STRESS FORCES

In this section, we introduce the connectivity and geometry of a structure with dihedral symmetry, and find the self-stress forces that equilibrate every node. The dihedral symmetry allows us to calculate symmetric state of self-stress by considering the equilibrium equations of only representative nodes.

2.1 Symmetry

We are considering prismatic and star tensegrity structures that have dihedral symmetry, denoted $\mathbf{D}_n$ (in the Schoenflies notation): there is a single major n-fold rotation (C_n^i) axis, which we assume is the vertical, z-axis, and n 2-fold rotation ($C_{2,j}$) axes perpendicular to this major axis. In total there are $2n$ symmetry operations. A structure has the same appearance before and after the transformation by applying any of these symmetry operations.

Consider a specific set of elements (nodes or members) of a structure with symmetry G. If one element in a set can be transformed to all of the other elements of that set by the symmetry
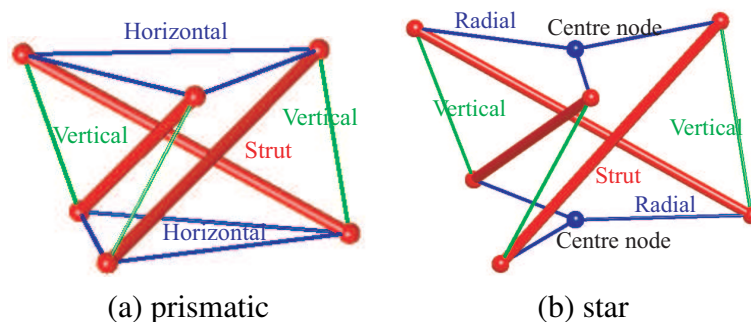

(a) prismatic   (b) star

Fig.1. Tensegrity structures that are of the same dihedral symmetry $\mathbf{D}_3$. The thick lines represent struts, and the thin lines represent cables. The horizontal cables in the prismatic structure are replaced by the radial cables connected to centre nodes in the star structure.
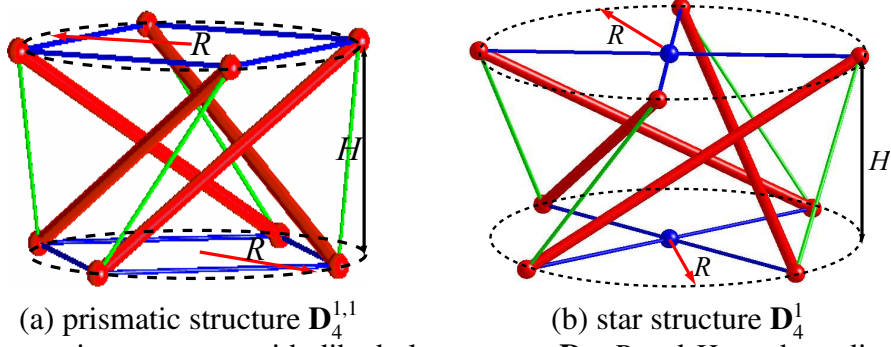
(a) prismatic structure $\mathbf{D}_4^{1,1}$ (b) star structure $\mathbf{D}_4^1$

Fig.2. The tensegrity structures with dihedral symmetry $\mathbf{D}_4$. R and H are the radius of the circle of boundary nodes and height of the structure, respectively.

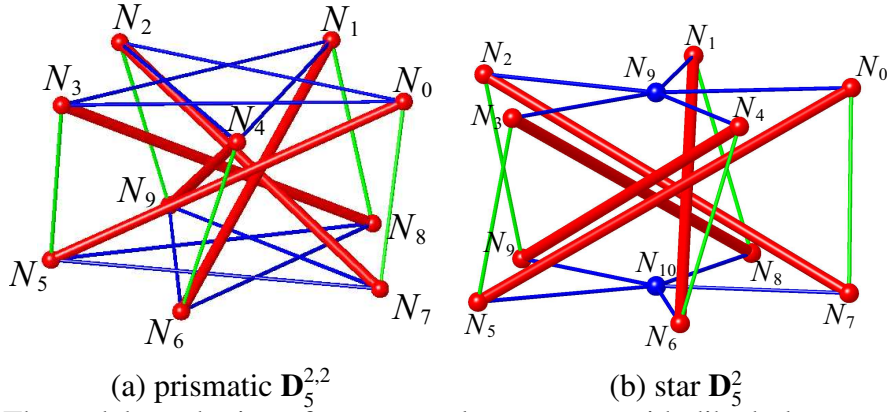(a) prismatic $\mathbf{D}_5^{2,2}$ (b) star $\mathbf{D}_5^2$

Fig.3. The nodal numbering of two example structures with dihedral symmetry $\mathbf{D}_5$.

operations in G, then this set of elements are said to belong to the same *orbit*. A structure can have several different orbits of elements.

In contrast to prismatic structures, which have only one orbit of nodes, there are two orbits of nodes in star structures—boundary nodes and centre nodes, as shown in Fig. 1. Thus, there are in total $2n$ nodes in a prismatic structure and $2n + 2$ nodes in a star structure. All the nodes are at $z = \pm H/2$ as shown in Fig. 2.

There are three orbits of members in a prismatic structure: horizontal cables, vertical cables and struts; and there are also three orbits of members in a star structure: instead of the horizontal cables, it has radial cables connected to the centre nodes. The members in each orbit have the same length and self-stress force, and therefore, the same force density, defined as ratio of self-stress force to member length. There are $2n$ horizontal cables in a prismatic structure and $2n$ radial cables, in a star structure, and n vertical cables and n struts in both.

2.2 Connectivity

The connectivities of prismatic and star structures are almost the same, except for those of horizontal cables and radial cables. By fixing the connectivity of struts, we use the notations $\mathbf{D}_n^{h,v}$ and $\mathbf{D}_n^v$ to describe the connectivity of a prismatic and a star structures with $\mathbf{D}_n$ symmetry, respectively. h and v here respectively denote the connectivity of the horizontal and vertical cables. The boundary nodes in the upper and lower circles are respectively numbered as $N_0, N_1 \ldots, N_{n-1}$ and $N_n, N_{n+1} \ldots, N_{2n-1}$, and the upper and lower centre nodes in the star structure are respectively numbered as N_{2n} and N_{2n+1}. We describe the connectivity of a reference node N_0 as follows — all other connections are then defined by the symmetry.
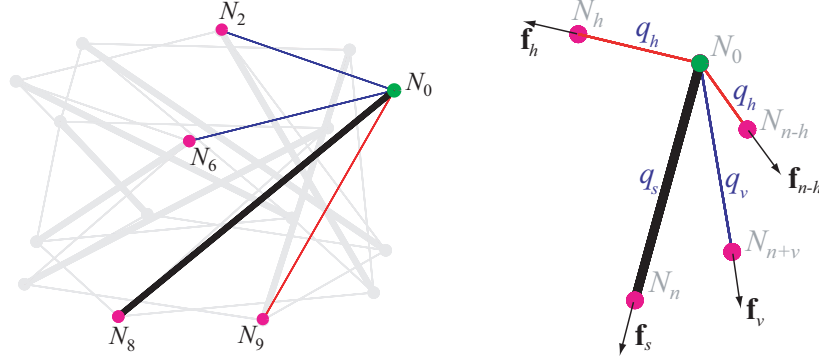
Fig.4. All nodes connected to a reference node N_0 of the prismatic structure $\mathbf{D}_8^{2,1}$. The three cable forces, $\mathbf{f}_h$, $\mathbf{f}_{n-h}$ and $\mathbf{f}_v$ are all tensile, and have a positive magnitude; the strut force $\mathbf{f}_s$ is compressive, and has a negative magnitude.

1. Without loss of generality, we assume that a strut connects node N_0 in the top plane to node N_n in the bottom plane.
2. A vertical cable connects node N_0 in the top plane to node N_{n+v} in the bottom plane. We restrict $1 \leq v \leq n/2$ (choosing $n/2 \leq v \leq n$ would give essentially the same set of structures, but in left-handed versions).
3. For the prismatic structure, a horizontal cable connects node N_0 to node N_h: symmetry also implies that a horizontal cable must also connect node N_0 to node N_{n-h}. We restrict $1 \leq h \leq n/2$.
4. For the star structure, a radial cable in the upper circle connects node N_0 to the centre node N_{2n}, and a radial cable in the lower circle connects node N_n to the centre node N_{2n+1}.

Fig. 3 shows the connectivities of the prismatic structure $\mathbf{D}_5^{2,2}$ and star structure $\mathbf{D}_5^2$.

2.3 Symmetric State of Self-stress Forces

Because of the high symmetry, we only need to consider the equilibrium of one reference node from each orbit, to find the symmetric state of self-stress forces. Thus we consider the equilibrium of one boundary node for prismatic structures, and one boundary and one centre node for star structures, in the absence of external forces. The force densities of the strut, vertical, horizontal and radial cables are respectively denoted as q_s, q_v, q_h and q_r.

2.3.1 Self-stresses of prismatic structures

Consider first the boundary nodes of a prismatic structure. Take one of them, for example node N_0 in the upper circle, see for example Fig. 4, as the *reference node* and let $\mathbf{x}_0$ denote its coordinates. The coordinates of the two boundary nodes in the upper circle, connected to the reference node by horizontal cables are denoted by $\mathbf{x}_h$ and $\mathbf{x}_{n-h}$, and those in the lower circle by strut and vertical cable are respectively denoted by $\mathbf{x}_s$ and $\mathbf{x}_v$.

When no external load is applied, the node N_0 should be equilibrated by the axial force vectors $\mathbf{f}_h$, $\mathbf{f}_{n-h}$, $\mathbf{f}_v$ and $\mathbf{f}_s$ of the cables and a strut as shown in Fig. 4, i.e.,

$$\mathbf{f}_h + \mathbf{f}_{n-h} + \mathbf{f}_s + \mathbf{f}_v = \mathbf{0}, \tag{1}$$

where

$$\begin{aligned} \mathbf{f}_h &= q_h(\mathbf{x}_h - \mathbf{x}_0), \quad \mathbf{f}_{n-h} = q_h(\mathbf{x}_{n-h} - \mathbf{x}_0), \\ \mathbf{f}_s &= q_s(\mathbf{x}_s - \mathbf{x}_0), \quad \mathbf{f}_v = q_v(\mathbf{x}_v - \mathbf{x}_0). \end{aligned} \tag{2}$$

Since all the boundary nodes belong to the same orbit, the coordinates of all other nodes can be obtained by a proper rotation matrix with the reference node. Hence, the self-equilibrium equation Eq. (1) can be assembled as follows with respect to $\mathbf{x}_0$

$$\tilde{\mathbf{S}}_{xyz}\mathbf{x}_0 = \mathbf{0}. \tag{3}$$

$\tilde{\mathbf{S}}_{xyz}$ is a block-diagonal matrix constructed from a 2-by-2 and a 1-by-1 sub-matrices on its leading diagonal. Both of these sub-matrices should be singular to allow the solution of Eq. (3) to give the position vector $\mathbf{x}_0$ of the reference node with non-trivial coordinates in three-dimensional space. Therefore, we have

$$q_v = -q_s \quad \text{and} \quad \frac{q_h}{q_v} = t = \frac{\sqrt{2 - 2C_v}}{2(1 - C_h)}. \tag{4}$$

2.3.2 Self-stresses of star structures

The self-stress forces of a star structure can be derived in a similar way as for the prismatic structure, except that we need to consider also the centre node, coordinates of which are denoted by $\mathbf{x}_c$, and the radial cables, force density of which is denoted by q_r.

The equilibrium of the reference node, in the absence of external force, is

$$q_s(\mathbf{x}_s - \mathbf{x}_0) + q_v(\mathbf{x}_v - \mathbf{x}_0) + q_r(\mathbf{x}_c - \mathbf{x}_0) = \mathbf{0}, \tag{5}$$

which can be rewritten as

$$\bar{\mathbf{E}}\mathbf{x}_0 + q_r\mathbf{x}_c = \mathbf{0}. \tag{6}$$

Ensuring non-trivial solutions for $\mathbf{x}_0$ leads to

$$q_v = -q_s, \quad \text{and} \quad \frac{q_r}{q_v} = \sqrt{2(1 - C_v)}. \tag{7}$$

Thus we have found force densities in the members that allow the star structure to be in self-equilibrium — equilibrium of the centre nodes is automatically satisfied. Note that the force density of vertical cables is identical to that of the prismatic structures.

3. SUPER STABILITY

In this section, we will find the stability conditions for prismatic and star structures. The force density matrix is critical to super stability of a tensegrity structure, and it can be studied in a much easier way making use of symmetry properties of the structures, as is shown in this section.

3.1 Force Density Matrix

The force density matrix $\mathbf{E}$ ($\in \mathfrak{R}^{2n \times 2n}$ for a prismatic structure and $\in \mathfrak{R}^{(2n+2) \times (2n+2)}$ for a star structure) is a symmetric matrix, defined by the force densities: Let $\mathcal{I}$ denote the set of members

connected to free node i, the (i, j)-component $\mathbf{E}_{(i,j)}$ of $\mathbf{E}$ is given as

$$
\mathbf{E}_{(i,j)} = \begin{cases} \sum_{k \in \mathcal{I}} q_k & \text{for } i = j, \\ -q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k, \\ 0 & \text{if nodes } i \text{ and } j \text{ are not connected,} \end{cases}
\tag{8}
$$

where q_k denotes the force density of member k.

The sufficient conditions for super stability of a tensegrity structure are ([3, 4])

1. The force density matrix has the minimum rank deficiency of four for a three-dimensional structure;
2. The force density matrix is positive semi-definite;
3. The member directions do not lie on a conic at infinity [3], or equivalently, the geometry matrix is full-rank [4].

Note that the first condition is also the necessary condition for non-degenerate structures [5], and the last condition is also the necessary condition for stable structures (with locally minimum strain energy). These two conditions are usually satisfied, and therefore, the second condition becomes crucial to super stability of a tensegrity structure.

One way of considering positive-definiteness of a matrix is to look at the signs of its eigenvalues: the force density matrix is positive semi-definite if all eigenvalues are equal to or greater than zero. Size of the force density matrix is proportional to the number of nodes, hence, verification of positive semi-definiteness of the matrix becomes difficult for the structures having large number of nodes, by applying conventional numerical methods such as eigenvalue analysis. Fortunately, the dihedral symmetry of the structures can be systematically utilized to analytically block-diagonalise the force density matrix into sub-matrices of only one or two dimensions as follows

$$
\tilde{\mathbf{E}} = \begin{bmatrix} \tilde{\mathbf{E}}^{A_1} & & & & & & & & \\ & \tilde{\mathbf{E}}^{A_2} & & & & & & & \\ & & (\tilde{\mathbf{E}}^{B_1}) & & & & \mathbf{O} & & \\ & & & (\tilde{\mathbf{E}}^{B_2}) & & & & & \\ & & & & \tilde{\mathbf{E}}^{E_1} & & & & \\ & & & & & \tilde{\mathbf{E}}^{E_1} & & & \\ & & \mathbf{O} & & & & \ddots & & \\ & & & & & & & \tilde{\mathbf{E}}^{E_p} & \\ & & & & & & & & \tilde{\mathbf{E}}^{E_p} \end{bmatrix},
\tag{9}
$$

where $p = (n - 1)/2$ for n odd and $p = (n - 2)/2$ for n even, and A_1, A_2, B_1 and B_2 refer to the one-dimensional representations in dihedral group, and E_k refer to the two-dimensional representations ([6, 7]). $\tilde{\mathbf{E}}^{A_1}$ and $\tilde{\mathbf{E}}^{A_2}$ are 1-by-1 matrices for the prismatic structure, and 2-by-2 matrices for star structure due to the existence of the centre nodes. The dimensions of other sub-matrices $\tilde{\mathbf{E}}^{\mu}$ are identical to those of the corresponding representations μ.

The sub-matrices for a prismatic structure are given as (see [8] for details)

$$
\tilde{\mathbf{E}}^{\mu} = q\mathbf{R}_0^{\mu} - q_h\mathbf{R}_h^{\mu} - q_h\mathbf{R}_{n-h}^{\mu} - q_s\mathbf{R}_n^{\mu} - q_v\mathbf{R}_{n+v}^{\mu},
\tag{10}
$$

and for a star structure (see [9] for details)

$$\tilde{\mathbf{E}}^\mu = q\mathbf{R}_0^\mu - q_r\mathbf{R}_r^\mu - q_s\mathbf{R}_n^\mu - q_v\mathbf{R}_{n+v}^\mu, \tag{11}$$

where $\mathbf{R}_i$ is the i-th irreducible representation matrix [6].

The number of sub-matrices corresponding to two-dimensional representations E_k increases along with the number of nodes, while the number of other sub-matrices will not change respectively for n odd or even. Therefore, we will only focus on the sub-matrices $\tilde{\mathbf{E}}^{E_k}$ corresponding to the two-dimensional representations E_k for the presentation of super stability condition.

3.2 Super Stability of Prismatic Structures

According to Eq. (10), $\tilde{\mathbf{E}}^{E_k}$ ($k = 1, \ldots, p$) of a prismatic structure $\mathbf{D}_n^{h,v}$ corresponding to the two-dimensional representations E_k are

$$\frac{1}{q_v}\tilde{\mathbf{E}}^{E_k} = \begin{bmatrix} 2t(1 - C_{hk}) + 1 - C_{vk} & -S_{vk} \\ -S_{vk} & 2t(1 - C_{hk}) - (1 - C_{vk}) \end{bmatrix}, \tag{12}$$

where C_{vk} and S_{vk} denote $\cos(2vk\pi/n)$ and $\sin(2vk\pi/n)$, respectively. The same formulation was derived in [2] making use of the special properties of the force density matrix as a circulant matrix. The eigenvalues of $\tilde{\mathbf{E}}^{E_k}$ are then calculated as

$$\frac{1}{q_v}\lambda_1^{E_k} = 2t(1 - C_{hk}) + \sqrt{2(1 - C_{vk})} > 0, \qquad \frac{1}{q_v}\lambda_2^{E_k} = 2t(1 - C_{hk}) - \sqrt{2(1 - C_{vk})}. \tag{13}$$

$\lambda_1^{E_k} > 0$ holds since $t > 0$, $1 - C_{hk} > 0$ and $1 - C_{vk} > 0$. For representation E_1, we know from Eq. (4) that $\lambda_2^{E_1} = 0$. To satisfy positive semi-definiteness and minimum rank deficiency of the force density matrix, $\lambda_2^{E_k}$ for $k > 1$ should be positive, for which it has been proved in [2] that this can be true if and only if $h = 1$; i.e., horizontal cables are connected to adjacent nodes.

3.3 Super Stability of Star Structures

According to Eq. (11), the sub-matrices $\tilde{\mathbf{E}}^{E_k}$ ($k = 1, \ldots, p$) of a star structure $\mathbf{D}_n^v$ corresponding to the two-dimensional representations E_k are given as

$$\tilde{\mathbf{E}}^{E_k} = \begin{bmatrix} q_r + q_v(1 - C_{kv}) & -q_vS_{kv} \\ -q_vS_{kv} & q_r - q_v(1 - C_{kv}) \end{bmatrix}, \tag{14}$$

the two eigenvalues of which are

$$\frac{1}{q_v}\lambda_1^{E_k} = \sqrt{2(1 - C_v)} + \sqrt{2(1 - C_{kv})} > 0, \qquad \frac{1}{q_v}\lambda_2^{E_k} = \sqrt{2(1 - C_v)} - \sqrt{2(1 - C_{kv})}. \tag{15}$$

For $k = 1$, we have $\lambda_2^{E_1} = 0$ as expected for the equilibrium condition. Thus, for $n = 3$ where $k > 1$ does not exist, the dihedral star tensegrity structure is super stable.

For $n > 4$, we must consider $\tilde{\mathbf{E}}^{E_k}$ for $k > 1$. For a super stable tensegrity structure, $\tilde{\mathbf{E}}^{E_k}$ for all $1 < k \le p$ must be positive definite; i.e., $\lambda_2^{E_k}$ must be positive; and hence, we require

$$C_{kv} > C_v, \quad \text{for all} \quad 1 < k \le (n - 1)/2. \tag{16}$$

Referring to [9], we have the following relation to ensure that the relation in Eq. (16) holds

$$v = \frac{n-1}{2}. \tag{17}$$

In other words, a dihedral star tensegrity structure is super stable if and only if it has odd number of struts (n odd), and the struts are as close to each other as possible ($v = (n-1)/2$).

4. DISCUSSION

In this study, we have shown the possibility of finding super stability condition for a whole class of structures, using their high symmetry. In particular, we have concentrated on prismatic and star tensegrity structures with dihedral symmetry:

- A prismatic structure is super stability as long as its horizontal cables are connected to the adjacent nodes;
- A star structure is super stable if and only if they have odd number of struts, while the struts are as close to each other as possible.

The methodology discussed in the study is also applicable to the structures with other point group symmetry.

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