TOPOLOGY OPTIMIZATION OF TRUSSES CONSISTING OF TRADITIONAL LAYOUTS

Ryo Watada[1)] **and Makoto Ohsaki**[2)]
1) Graduate Student, Kyoto University, Kyoto, Japan
E-mail: is.watada@archi.kyoto-u.ac.jp
2) Associate Professor, Kyoto University, Kyoto, Japan
E-mail: ohsaki@archi.kyoto-u.ac.jp

ABSTRACT

A linear Mixed Integer Programming (MIP) approach is presented for topology optimization of trusses with discrete variables. The problem with stress constraints is reformulated to linear MIP problem. The objective function is the total structural volume. It is shown in the numerical examples that an optimal combination of the traditional layouts of plane and space trusses can be successfully found by solving linear 0-1 MIP problems. A systematic approach is developed for generating the constraints on the 0-1 variables for selection of members representing the traditional layouts. The effectiveness of the proposed method is demonstrated in the numerical examples.

1. INTRODUCTION

Truss topology optimization problem is one of the traditional problems in structural optimization, and many methodologies have been developed. The most popular approach is the so called ground structure approach, where the cross-sectional areas are the continuous variables, and the unnecessary members are removed after optimization from the highly connected ground structure [1].

However, in practical situation, the cross-sectional properties are selected from the discrete list of available sections. Therefore, the problem turns out to be a Mixed Integer Programming (MIP) problem [2]. Many heuristic approaches including genetic algorithm and simulated annealing have been developed for this problem to find approximate optimal solutions within practically acceptable computational cost.

Another aspect of truss design is that there are several *traditional* layouts such as Schwedler-dome, Lamella-dome and Warren truss. In this study, we optimize a truss as a combination of traditional layouts under stress constraints to explore the efficiency of those layouts [3].

2. MIP FORMULATION FOR STRESS CONSTRAINTS

Consider a pin-jointed truss, and let A_i and L_i denote the cross-sectional area and length of the ith member, respectively. A_i is selected from the set of available sections as $A_i \in \{0, a_i^1, \ldots, a_i^{r_i}\}$ $(i = 1, \ldots, m)$, where m is the number of members, and r_i is the number of available sections for the ith member. The nodal displacement vector against the static loads $\mathbf{P}$ is denoted by $\mathbf{U}$. The member elongation δ_i is written with respect to $\mathbf{U}$ as,

$$\delta_i = \mathbf{b}_i^\top \mathbf{U}, \quad (i = 1, \ldots, m) \tag{1}$$

where $\mathbf{b}_i$ is a constant vector. Then the axial force vector $\mathbf{N} = (N_1, \ldots, N_m)$ is given with the elastic modulus E as

$$N_i = \frac{A_i E}{L_i} \mathbf{b}_i^\top \mathbf{U}, \quad (i = 1, \ldots, m) \tag{2}$$

Let σ_i^{U} and σ_i^{L} denote the upper and lower bounds for the stress of the ith member. The equilibrium matrix is denoted by $\mathbf{D}$, for which the ith column is equal to $\mathbf{b}_i$. Then the topology optimization problem for minimizing the total structural volume under stress constraints is formulated as

$$\text{Minimize} \quad V = \sum_{i=1}^{m} A_i L_i \tag{3a}$$

$$\text{subject to} \quad \mathbf{D}\mathbf{N} = \mathbf{P} \tag{3b}$$

$$A_i \sigma_i^{\mathrm{L}} \leq N_i \leq A_i \sigma_i^{\mathrm{U}}, \quad (i = 1, \ldots, m) \tag{3c}$$

$$N_i = A_i \frac{E}{L_i} \mathbf{b}_i^\top \mathbf{U}, \quad (i = 1, \ldots, m) \tag{3d}$$

$$A_i \in \{0, a_i^1, \ldots, a_i^{r_i}\}, \quad (i = 1, \ldots, m) \tag{3e}$$

A 0-1 variable x_i^k is defined as

$$x_i^k = \begin{cases} 1 & \text{if } A_i = a_i^k \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Then A_i is written with a_i^k and x_i^k as

$$A_i = \sum_{k=1}^{r_i} x_i^k a_i^k, \tag{5a}$$

$$\sum_{k=1}^{r_i} x_i^k \leq 1 \tag{5b}$$

An auxiliary variable s_i^k is also used for the axial force as

$$s_i^k = x_i^k a_i^k \frac{E}{L_i} \mathbf{b}_i^\top \mathbf{U}, \tag{6a}$$

$$N_i = \sum_{i=1}^{r_i} s_i^k \tag{6b}$$

Note that (6a) is nonlinear with respect to x_i^k and $\mathbf{U}$.

Let $\mathbf{U}^{\min} = (U_1^{\min}, \dots, U_n^{\min})^\top$ and $\mathbf{U}^{\max} = (U_1^{\max}, \dots, U_n^{\max})^\top$ denote the vectors of lower and upper bounds of displacements, where n is the number of degrees of freedom. $c^{\min}$ and $c^{\max}$ are defined as sufficiently small lower bound and large upper bound of member elongation corresponding to the displacements satisfying $\mathbf{U}^{\min} \leq \mathbf{U} \leq \mathbf{U}^{\max}$. Then the nonlinear relation (6a) is converted to the following pair of inequalities [2, 4]:

$$(1 - x_i^k)c_i^{\min} \leq a_i^k \frac{E}{l_i} \mathbf{b}_i^\top \mathbf{U} - s_i^k \leq (1 - x_i^k)c_i^{\max} \tag{7}$$

It is seen from (7) that if $x_i^k = 1$, then the left-hand-side and right-hand-side terms vanish and the relation (6a) is satisfied. On the other hand, if $x_i^k = 0$, then the inequalities are satisfied by any values of $\mathbf{U}$ satisfying $\mathbf{U}^{\min} \leq \mathbf{U} \leq \mathbf{U}^{\max}$ and s_i^k; i.e., no constraint is given for s_i^k.

The objective function is the total structural volume V. Then the topology optimization problem is formulated as

$$\text{Minimize} \quad V = \sum_{i=1}^{m} \sum_{k=1}^{r_i} x_i^k a_i^k L_i \tag{8a}$$

$$\text{subject to} \quad \mathbf{DN} = \mathbf{P} \tag{8b}$$

$$x_i^k a_i^k \sigma_i^{\mathrm{L}} \leq s_i^k \leq x_i^k a_i^k \sigma_i^{\mathrm{U}}, \quad (i = 1, \dots, m;\ k = 1, \dots, r_i) \tag{8c}$$

$$U_j^{\min} \leq U_j \leq U_j^{\max}, \quad (j = 1, \dots, n) \tag{8d}$$

$$x_i^k c_i^{\min} \leq s_i^k \leq x_i^k c_i^{\max}, \quad (i = 1, \dots, m;\ k = 1, \dots, r_i) \tag{8e}$$

$$(1 - x_i^k)c_i^{\min} \leq a_i^k \frac{E}{L_i} \mathbf{b}_i^\top \mathbf{U} - s_i^k \leq (1 - x_i^k)c_i^{\max}, \quad (i = 1, \dots, m;\ k = 1, \dots, r_i) \tag{8f}$$

$$\sum_{k=1}^{r_i} x_i^k \leq 1, \quad (i = 1, \dots, m) \tag{8g}$$

$$x_i^k \in \{0, 1\}, \quad (i = 1, \dots, m;\ k = 1, \dots, r_i) \tag{8h}$$

3. DECISION TREE FOR GENERATING AUXILIARY 0-1 VARIABLES AND CONSTRAINTS

The auxiliary variables and constraints are generated using the decision tree as shown in Fig. 1. For example, the conditions for selecting the members of a truss is defined by the following node i of the tree:

$$\left\{ \begin{array}{c} (1, 2) \\ [3, 4, \phi_1]_1 \\ [5, 6, 7, \phi_2, \phi_3]_2 \end{array} \right\}^i \tag{9}$$

Here, the first condition $(j_1, \cdots, j_n)$ denoted by EC^i means that all members in $(\cdots)$ are to be selected. On the other hand, the kth condition $[j_1, \cdots, j_n]_m$ of node i denoted by CC_k^i means that the maximum of m members are selected from $[\cdots]$, where ϕ_j is a dummy member. The

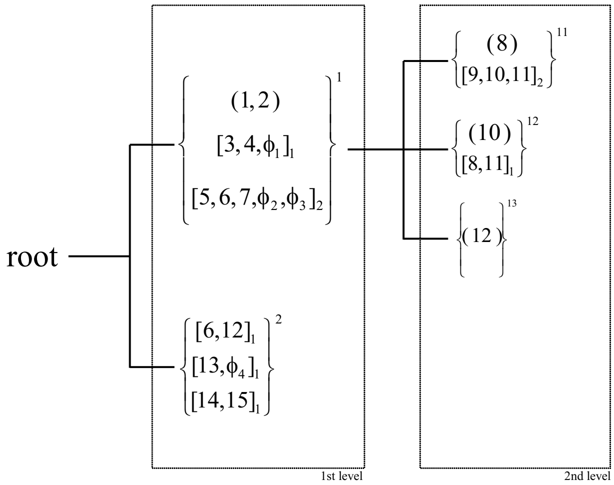


Fig.1. An example of decision tree.

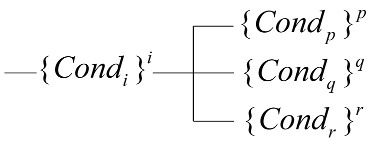


Fig.2. Relation between nodes of a decision tree.

set of conditions at node i of the decision tree is denoted by $Cond_i$. Suppose the set of members $\mathcal{S}_i^E$ are selected by EC^i, and the set of members and the number of members to be selected in CC_k^i are denoted by $\mathcal{S}_{ik}^C$ and m_{ik}, respectively. Then $Cond_i$ is given as

$$Cond_i = \left\{ EC^i, CC_k^i \mid k = 1, \dots, d_i \right\} \tag{10a}$$

$$EC^i \ : \ \left(\{ \ j \mid j \in \mathcal{S}_i^E \} \right) \tag{10b}$$

$$CC_k^i \ : \ \left[\{ \ j \mid j \in \mathcal{S}_{ik}^C \} \right]_{m_{ik}} \tag{10c}$$

Consider the case where nodes p, q, and r exist below node i as shown in Fig. 2. Nodes p, q, and r are the child nodes of node i, and node i is the parent node of p, q, and r. Then the following condition is satisfied:

- If $Cond_i$ is active, then one of $Cond_p$, $Cond_q$, and $Cond_r$ is active; otherwise, none of $Cond_p$, $Cond_q$, and $Cond_r$ is active.

When r is a dummy node, then selection of $Cond_i$ and $Cond_r$ leads to the selection of $Cond_i$ only. A 0-1 variable y_j is introduced for node j as

$$y_j = \begin{cases} 1 & \text{if } \ Cond_j \ \text{ is active} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Hence, the relation in Fig. 2 is written as

$$y_i = y_p + y_q + y_r \tag{12}$$

Each set of selected members of a truss is defined by the path of the decision tree from the *root* without parent to the *leaf* without child.

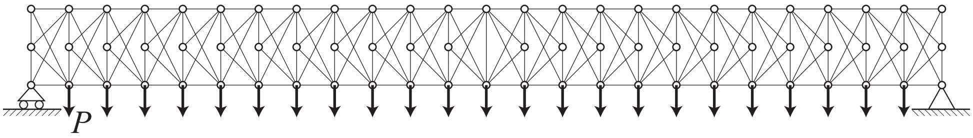


Fig.3. A bridge-type plane truss.

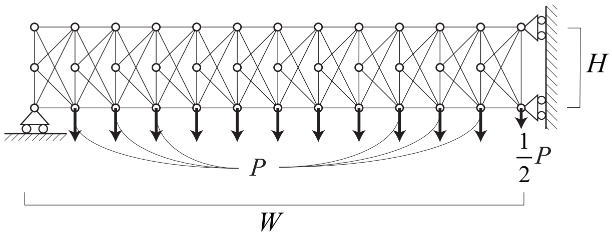


Fig.4. Half part of the bridge-type plane truss.

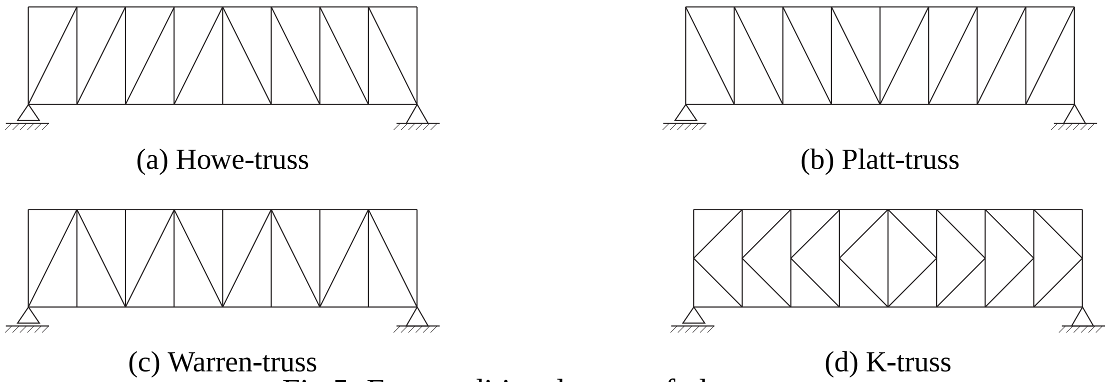


Fig.5. Four traditional types of plane truss.

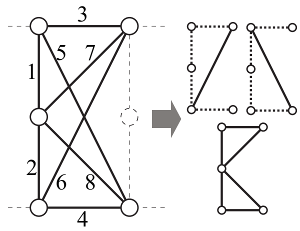


Fig.6. Selection of layouts.

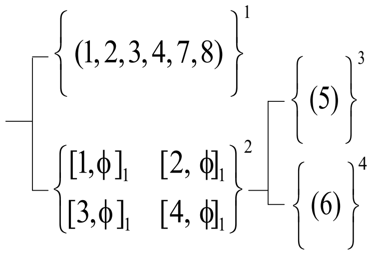
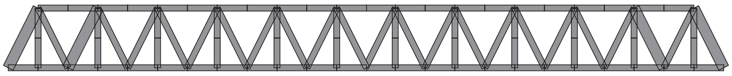


Fig.7. Decision tree for the bridge-type truss.

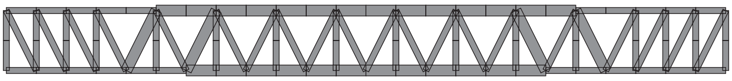
4. NUMERICAL EXAMPLES

Consider a plane truss as shown in Fig. 3, which has many members as an assemblage of the four traditional layouts; namely, Howe-truss, Platt-truss, Warren-truss and K-truss, as shown in Fig. 5(a)–(d), respectively [5]. Uniform loads P are applied at the lower nodes. Incorporating the symmetry conditions of the loads, the topology of the truss is also assumed to be symmetric; hence, one of the half parts as shown in Fig. 4 is to be optimized. We use the optimization library CPLEX 10.2 [6] for solving the MIP. The computation is carried out on the PC with Intel Xeon CPU 3.40GHz, 2.00GB RAM.

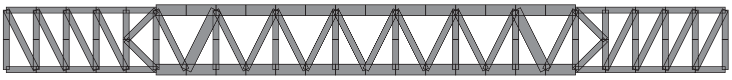
The members are appropriately divided into groups so that one of the four types is selected. For this purpose, the units are classified to three groups as shown in Fig. 6. The parameters are $h = 2.0$, $w = 12.0$, $E = 2.0$ and $\sigma_i^{\mathrm{U}} = 0.003$ for all members. The lower-bound stress is given



(a) Case 1: CPU = 3500, V = 65.410.

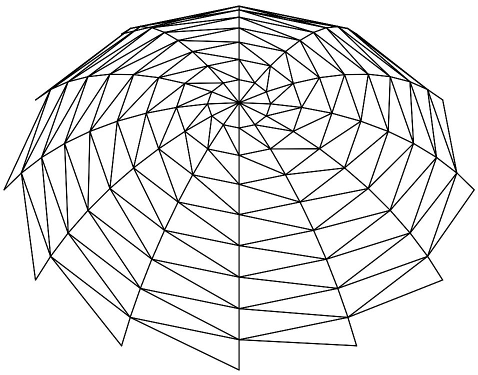


(b) Case 2: CPU = 960, V = 80.810.

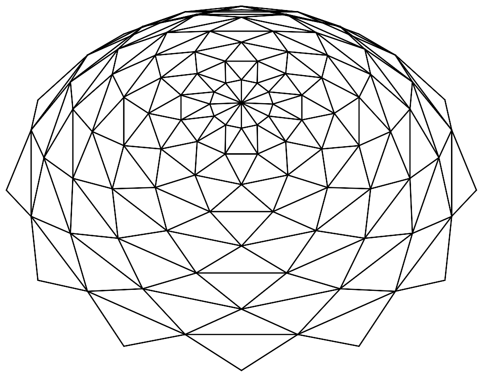


(c) Case 3: CPU = 380, V = 82.414.

Fig.8. Optimal topologies and cross-sectional areas of the plane truss.



(a) Schwedler-dome



(b) Lamella-dome

Fig.9. Two traditional types of space truss.

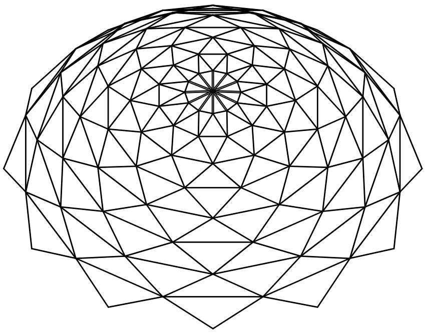
by

$$\sigma_i^{\mathrm{L}} = -\frac{L_{\min}^2}{L_i^2}\sigma_i^{\mathrm{U}} \tag{13}$$

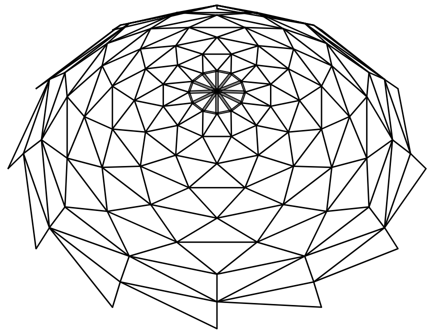
to incorporate member buckling, where $L_{\min}$ is the length of the shortest member. The cross-sectional areas are chosen from the list $\{0.0, 1.0, 1.8\}$. In order to improve the computational efficiency, the variable s_i^k is scaled to s_i^{k*} as

$$s_i^{k*} = \frac{L_i}{Ea_i^k}s_i^k \tag{14}$$

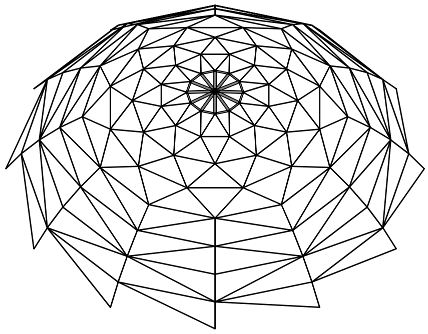
The optimal solutions are found for Cases 1–3 with three different load magnitudes $P = 0.00007$, 0.00012 and 0.00014, respectively. The optimal topologies are as shown in Fig. 8(a)–(c), where the width of each member is proportional to its cross-sectional area, and the CPU time (sec.) and the optimal objective value are shown in each figure. As is seen, the Warren-truss tends to be selected for a smaller load, while Howe-truss is selected for a larger load. Note that CPU time strongly depends on the load magnitude; i.e. computational cost is smaller for larger load, because the number of admissible set of cross-sectional areas decreases as the load is increased. We next optimize a dome truss as an assemblage of the Schwedler-dome and Lamella-dome as shown in Fig. 9(a) and (b), respectively, subjected to the vertical load P at the top node. The cross-sectional areas are selected from the list $\{0.0, 1.0, 3.0, 5.0\}$, and we consider Cases



(a) Case 1: V = 1404.3.



(b) Case 2: V = 1441.1.



(c) Case 3: V = 1474.4.

Fig.10. Optimal topologies and cross-sectional areas of the space dome truss.

1–3 with load values P = 0.001, 0.002, and 0.0025, respectively. The optimal topologies are as shown in Fig. 10(a)–(d). As is seen, the optimal truss is a Lamella-dome for Case 1 with smallest load, while Schwedler-dome dominates for larger load magnitudes. CPU times (sec.) for Cases 1-3 are 12, 688, and 680, respectively.

5. CONCLUSIONS

An approach based on the mixed integer programming has been presented for topology optimization of trusses as an assemblage of traditional layouts. The cross-sectional areas are selected from the list of available values. The problem under stress constraints has been reformulated to a MIP problem with 0-1 variables. A systematic procedure has been developed to generate the constraints and auxiliary variables for selecting the layouts from the predefined list. It has been shown that optimal assembly of the layouts can be successfully obtained by the proposed approach. The computational cost strongly depends on the ratio of the load and cross-sectional area.

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