Optimization Methods for Force and Shape Design of Tensegrity Structures

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Abstract
This paper presents three optimization approaches for force and shape design of tensegrity structures. The first approach is to find the self-equilibrated configuration of a tensegrity structure, by minimizing the difference of strain energy between cables and struts; the second method is to find self-equilibrated configurations for the structures modeled as directed graphs, where the deviation of member forces from target values is to be minimized; and the third method is to find the optimal distribution of member forces so as to let the structure have maximum stiffness as well as uniform member forces.

Keywords: Tensegrity; Shape design; Force design; Optimization.

1. Introduction

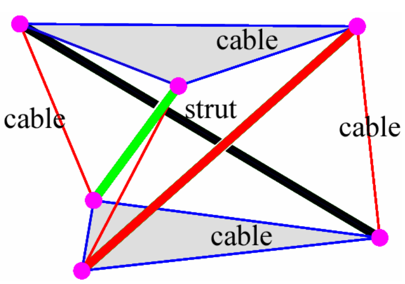
Shape and member forces of a tensegrity structure are highly interdependent on each other, in the aspects of self-equilibrium as well as stability. Shape design and force design, for the determination of self-equilibrated and stable configuration for a tensegrity structure, are the two major subjects in the design of tensegrity structures. In this paper, we introduce some of our attempts to tackle these problems making use of the well-developed optimization techniques.

Tensegrity structure is a kind of prestressed pin-jointed structure. It consists of members either in compression or in tension. The term '*tensegrity*' is a contraction of 'tensional' and 'integrity', referring to the integrity of stable structures based on a synergy between the balanced compressive and tensile members[1]. The members in compression and in tension are respectively called *struts* and *cables*, and they are respectively shown in thick and thin lines in the figures in the paper. A tensegrity structure can be free-standing without any fixed nodes, e.g. the prismatic structure in Figure.1(a), or can be attached to supports, e.g. the tensegrity dome in Figure.1(b). Principles of tensegrity structures have been successfully applied in many different disciplines, such as architecture, mechanical engineering, bio-medical engineering, and mathematics.

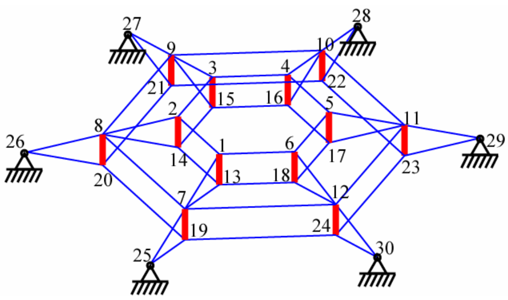
Among the many distinct properties of tensegrity structures compared to conventional structural forms, self-equilibrium and stabilizing effect by the prestress are their spirits. Configuration and distribution of member forces of a tensegrity structures are the major factors that have significant influence on these two properties. Hence, *shape design*, which is to determine the self-equilibrated configuration, and *force design*, which is to determine the distribution of member forces, are of great importance in the design of tensegrity structures. These two design problems can be separately dealt with, and can be combined together, subjected to given conditions.

Following this introductory section, we will discuss three different optimization problems for the shape design and force design problems of tensegrity structures:

(1) Section 2 is to have direct control over the *magnitudes of forces* of some members, by solving the problem of minimizing the difference between strain energy of cables and struts.



(a) Prismatic structure



(b) Tensegrity dome

Figure 1. Examples of tensegrity structure. (a) is free-standing, and (b) is attached to supports.

(2) The approach in Section 3 is for the control of *member directions*, where tensegrity structures are modeled as directed graphs, and their self-equilibrated configurations are determined by minimizing the deviation of member forces from target values.

(3) Section 4 deals with the structures with given configurations and more than one mode of member forces; our objective is to find the Pareto optimal solutions for force distribution leading to strongest structures as well as minimum deviation of member forces in the same group of members.

These studies are briefly discussed and concluded in Section 5.

For the sake of simplification, the following assumptions are adopted for all cases considered in the study:

(a) The members are straight and are connected by pin-joints.
(b) Neither self-weight nor external load is considered.
(c) Member failure, such as yielding or buckling, is not considered.
(d) Topology of the structure is assumed to be given.

Moreover, n, m and d are global variables in the paper, denoting the numbers of nodes and members, and dimensions of the structure, respectively.

2. Self-equilibrated Configuration by Strain Energy Difference

This section presents an optimization problem, considering the minimum difference of strain energy between cables and struts, for the determination of self-equilibrated configurations of tensegrity structures. Some member forces can be directly specified by the designers satisfying the self-equilibrium equations. The proposed method is an extension of the same idea for cable nets to the cases of tensegrity structures.

Let l_i^0 and l_i respectively denote the lengths of member i at the unstressed and stressed states, and A_i is its cross-sectional area. E is Young's modulus, which is assumed to be the same for all members. The force s_i of member i can be defined as $s_i = EA_i\varepsilon_i$. Hence, if we assume that the absolute value of the strain is fixed at $\bar{\varepsilon}$, and s_i can be adjusted by varying A_i.

Let $\mathbf{v}_k = (v_k^x, v_k^y, v_k^z)^{\mathrm{T}}$ denote the force vector of member k. The axial force vector for structure is combined as

$$\mathbf{v} = (\mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}}, ..., \mathbf{v}_m^{\mathrm{T}})^{\mathrm{T}} \tag{1}$$

$\mathbf{B}_i$ defines the topology of member i connected to free nodes, l_i can be computed as

$$l_i = \| \mathbf{B}_i \mathbf{x} - \mathbf{d}_i \| \tag{2}$$

where $\mathbf{x}$ is the coordinate vector and $\mathbf{d}_i$ is the coordinate vector of the fixed node connected to member i. Combining $\mathbf{B}_i$ for each member as $\mathbf{B} = [\mathbf{B}_1^{\mathrm{T}}, ..., \mathbf{B}_m^{\mathrm{T}}]^{\mathrm{T}}$, self-equilibrium equation of the structure can be written as

$$\mathbf{Bv} = \mathbf{0} \tag{3}$$

Note that the variables in the proposed approach are not the scalar values of the member forces, but the components of the force vectors. The symmetry constraint can be written as follows with respect to the force vector $\mathbf{v}$

$$\mathbf{Sv} = \mathbf{0} \tag{4}$$

Eqs. (3) and (4) lead to the following linear equation with respect to $\mathbf{v}$

$$\mathbf{Cv} = \mathbf{0}, \quad \text{where } \mathbf{C} = \left[\mathbf{B}^{\mathrm{T}}, \mathbf{S}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{5}$$

From which, there are $\beta = 3m - \mathrm{rank}(\mathbf{C})$ independent member forces in total that can be arbitrarily specified.

To determine the configuration of a tensegrity structure with some given member forces, in terms of cross-sectional areas, we consider the following optimization problem

$$\text{(A)} \qquad \text{Minimize } \sum_{\text{cables}} EA_i\bar{\varepsilon}^2 l_i^0 / 2 - \sum_{\text{struts}} EA_i\bar{\varepsilon}^2 l_i^0 / 2$$
$$\text{Subject to } (1+\bar{\varepsilon}) l_i^0 = l_i = \| \mathbf{B}_i \mathbf{x} - \mathbf{d}_i \|$$

The variables involved in problem (A) are the nodal coordinates $\mathbf{x}$ of the free nodes. It was proved in [2] that the optimal solution of problem (A) corresponds to the structure at the state of self-equilibrium. Hence, the self-equilibrated configuration can be determined by solving the problem (A). It should be noted that the constraint may not be differentiable at some points at the boundary of the feasible region. Therefore, difficulty arises in applying non-linear programming to this problem, because differential (sensitivity) coefficients are necessary but they are unavailable at these points.

The difficulty in nondifferentiability of the constraints can be avoided, by using the primal-dual interior-point methods, because the solution monotonically converges at the interior point of the feasible region. For this purpose, the constraint in problem (A) is rewritten as follows to define the feasible region F(A)

$$F(A) = (1+\bar{\varepsilon})l_i^0 \geq l_i \quad \text{for cables}$$
$$(1+\bar{\varepsilon})l_i^0 \leq l_i \quad \text{for struts} \tag{6}$$

The process of determining self-equilibrated configurations of tensegrity structures by solving problem (A) within the feasible region defined in (6) is summarized as follows:

Form-finding algorithm 1:

Step 0: Specify topology, assign member type (cable or strut) to each member, and define constraints.

Step 1: Specify cross-sectional areas for β members.

Step 2: Assign $\bar{\varepsilon}$, and compute A_i for the remaining $m - \beta$ each member.

Step 3: Use primal-dual interior method to solve problem (A) to determine the configuration in terms of nodal coordinates **x**.

For the cable nets, which are special cases of tensegrity structures with all members in tension, the problem (A) is convex, and therefore, the primal-dual interior method converges at the global optimal solution corresponding to the self-equilibrated configuration[3]. Moreover, numerical examples in [2] showed that this approach can find the configurations precisely for tensegrity structures in some cases, however, this cannot be guaranteed since the problem (A) for them turns out to be non-convex.

3. Self-equilibrated Configuration in Direct Graph

Tensegrity structures can be easily modeled as directed graphs, since their members are straight and carry only axial forces, either tension or compression. The nodes and members of a structure correspond to the vertices and edges of a graph. The authors have proposed a direct approach based on this basic idea to have direct control over member directions of the structures, by consecutively specifying independent components of member force and coordinates[4]. However, the number of variables that need to be specified may turn out to be much greater than expected, especially for complicated structures. This motivates us to consider the optimization problem presented in this section to help designers out from the tedious tasks, while reserving advantages of the direct approach.

3.1 Formulations

The direction $\mathbf{v}_k$ of a member is defined as shown in Figure. 2. Suppose that member k connects the nodes i and j where $i < j$. We define that $\mathbf{v}_k$ is directed from node i to node j if the member is a cable, as shown in Figure. 2(a), and from node j to i if a strut, as shown in Figure. 2(b).

Let $\mathbf{d}_k = [d_k^x, d_k^y, d_k^z]^{\mathrm{T}}$ denote the coordinate difference vector of member k. It is directed from node i to node j for $i < j$. Since $\mathbf{d}_k$ and $\mathbf{v}_k$ are parallel to each other, the following relation is obtained from the condition that the vector product of the two vectors should vanish:

$$\left[\mathrm{diag}(\overline{\mathbf{T}}\mathbf{v}_k) - \mathrm{diag}(\mathbf{v}_k)\overline{\mathbf{T}}\right]\mathbf{d}_k = \mathbf{0}, \qquad \text{where } \overline{\mathbf{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{7}$$

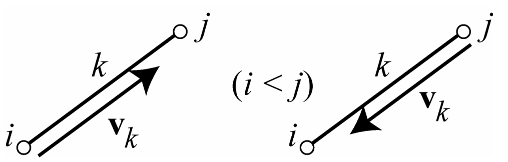
By assembling (7) through all members, and by substituting $\mathbf{d}_k$ by the nodal coordinate vector $\mathbf{x}$ for the whole structure, we can obtain the following constraints with respect to $\mathbf{x}$:

$$\mathbf{Fx} = \mathbf{0} \tag{8}$$

Note that $\mathbf{F}$ is a function of $\mathbf{v}$.

The constraint on $\mathbf{v}$ in (5) has to be strictly satisfied, and is so called *hard constraint*. Since $\mathbf{C}$ does not usually have full rank, some components of $\mathbf{v}$ can be arbitrarily specified as discussed in [4]. However, for the structure with large number of members, it may become much difficult to assign appropriate values to all these unknown parameters to obtain an expected shape. This motivates us to present an effective approach to deal with this problem taking advantage of optimization techniques.

Let $\bar{\mathbf{v}}$ denote the target value of $\mathbf{v}$. The linear relations with respect to the components of $\mathbf{v}$ that should be preferably and approximately satisfied, called *soft constraints*, are assigned as



(a) Member direction in tension (b) Member direction in compression

Figure 2. Definition of the direction of member force vector.

$$\mathbf{Rv} = \mathbf{0} \tag{9}$$

Note that symmetry properties can also be considered as soft constraints, in stead of hard constraints adopted in the paper. Hence the objective function $E(\mathbf{v})$ to be minimized is defined as

$$E(\mathbf{v}) = \frac{1}{2}(\mathbf{v} - \overline{\mathbf{v}})^{\mathrm{T}} \mathbf{W}^{\mathrm{I}} (\mathbf{v} - \overline{\mathbf{v}}) + \frac{1}{2}(\mathbf{Rv})^{\mathrm{T}} \mathbf{W}^{\mathrm{II}} (\mathbf{Rv}) \tag{10}$$

where $\mathbf{W}^{\mathrm{I}}$ and $\mathbf{W}^{\mathrm{II}}$ are diagonal weighting matrices that have the weight coefficients w_i^{I} and w_i^{II} at the ith diagonal entries, respectively. By increasing w_i^{I} and w_i^{II}, the deviation of the ith component v_i of $\mathbf{v}$ from its target value and the error of the ith soft constraint, respectively, can be reduced.

The problem is formulated as

(B) Minimize $E(\mathbf{v})$

Subject to $\mathbf{Cv} = \mathbf{0}$

The problem (B) is a convex quadratic programming problem with linear equality constraints, and therefore, it can be easily solved using the Lagrangian multiplier approach formulated as

$$L(\mathbf{v}, \boldsymbol{\mu}) = E(\mathbf{v}) + \boldsymbol{\mu}^{\mathrm{T}} \mathbf{Cv} \tag{11}$$

where $\boldsymbol{\mu}$ denotes the vector of Lagrange multipliers. The stationary conditions of L with respect to $\mathbf{v}$ and $\boldsymbol{\mu}$ lead to

$$\begin{bmatrix} \mathbf{W}^{\mathrm{I}} + \mathbf{R}^{\mathrm{T}} \mathbf{W}^{\mathrm{II}} \mathbf{R} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{O} \end{bmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{W}^{\mathrm{I}} \overline{\mathbf{v}} \\ \mathbf{0} \end{pmatrix} \tag{12}$$

where $\mathbf{O}$ is a null matrix. $\mathbf{v}$ and $\boldsymbol{\mu}$ can be obtained by solving the linear equations (12). Note that sufficient number of force components is needed in the objective function to prevent singularity. The whole process of determination of self-equilibrated configurations for tensegrity structures by solving the problem (B) is summarized as follows:

Form-finding algorithm 2:

Step 0: Specify the topology of the tensegrity structure.

Step 1: Construct the matrix $\mathbf{B}$ and specify the hard constraints. Assemble them to obtain the linear constraints $\mathbf{Cv} = \mathbf{0}$. (Eq. (5))

Step 2: Assign the target force vector $\overline{\mathbf{v}}$, the soft constraints (9), and the weights w_i^{I} and w_i^{II}, to define the objective function $E(\mathbf{v})$. (Eq. (10))

Step 3: Solve (12) for $\mathbf{v}$ and $\boldsymbol{\mu}$.

Step 4: Construct $\mathbf{F}$ from $\mathbf{v}$ to formulate the self-equilibrium equation $\mathbf{Fx} = \mathbf{0}$ with respect to the nodal coordinates $\mathbf{x}$. (Eq. (8))

Step 5: Compute the rank r of $\mathbf{F}$ and specify $\eta = dn - r$ independent components of nodal coordinates to determine configuration of the structure in terms of $\mathbf{x}$ [4].

3.2 Numerical Example: *two-stage tensegrity structure*

The capability of the proposed method for controlling the shapes and forces of tensegrity structures is demonstrated by a two-stage tensegrity. The structure as shown in Figure. 3 consists of 6 struts and 24 cables. To find appropriate target values of the force components as in Step 2 in Algorithm 2, the structure is constructed from the icosahedron as shown in Figure. 3(a). Note, in Figure. 3(b), that most of the cable members are omitted for clarity.

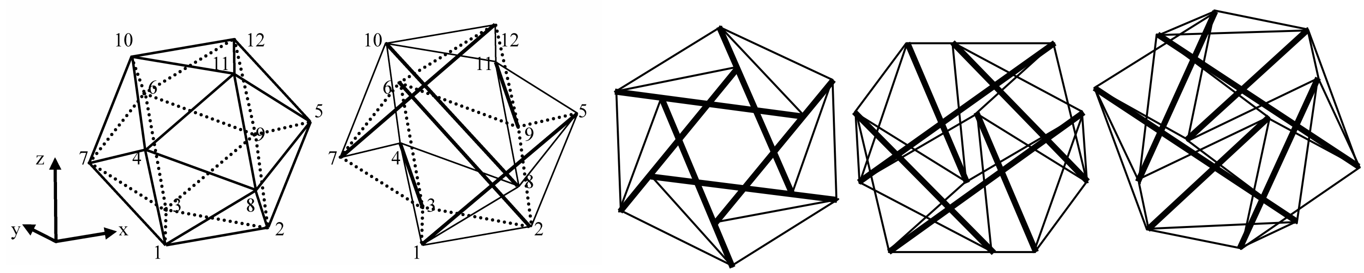
Topology of the structure defined by the relation between the node numbers and member numbers are listed in Table 1, where the number at the jth column of the ith row indicates the member number that connects nodes i and j, and an italic number indicates a strut. In the example, rotational symmetry about z-axis is assigned as hard constraints.

From the equilibrium at each node of the structure in Figure. 3(b), the target force vectors of the cable members are listed in Table 2. The objective function is given as

$$E_1(\mathbf{v}) = \frac{1}{2} \sum_{\text{cables}} (\mathbf{v}_i - \overline{\mathbf{v}}_i)^{\mathrm{T}} (\mathbf{v}_i - \overline{\mathbf{v}}_i) \tag{13}$$

where no soft constraint is considered. Solving the problem (B) in (12) with the objective function in (13), the force vectors are derived as listed in Table 3. It can be observed in Table 3 that the final solution for the force vectors is only slightly different from their target values obtained from similarity with member directions of the icosahedron. This shows high similarity of the icosahedron and the structure considered here.

Taking symmetry conditions in (4) as hard constraints, we have $\mathrm{rank}(\mathbf{F}) = 32$. Since the number of nodes n is 12, there are four coordinate components in total that can be arbitrarily specified. Suppose that they are given as $(x_1, y_1, z_1, x_2) = (0, 0, 0, 1.05)$, self-equilibrated configuration of the structure is obtained as shown in Figure. 3(c)-(e).



(a) Icosahedron (b) Example structure (c) Top view (d) Side view (e) Bird view

Figure 3. Similarity between the icosahedron and a tensegrity.

Table 1. Topology of the tensegrity defined by the relation between the node numbers and member numbers

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	·	1	3	4	*9*	·	·	10	·	·	·	·
2	1	·	2	·	5	*7*	·	·	11	·	·	·
3	3	2	·	*8*	·	6	12	·	·	·	·	·
4	4	·	*8*	·	·	·	25	28	·	22	·	·
5	*9*	5	·	·	·	·	·	26	29	·	23	·
6	·	*7*	6	·	·	·	30	·	27	·	·	24
7	·	·	12	25	·	30	·	·	·	13	·	*21*
8	10	·	·	28	26	·	·	·	·	*19*	14	·
9	·	11	·	·	29	27	·	·	·	·	*20*	15
10	·	·	·	22	·	·	13	*19*	·	·	16	18
11	·	·	·	·	23	·	·	14	*20*	16	·	17
12	·	·	·	·	·	24	*21*	·	15	18	17	·

Table 2. Target values of force vectors of the cable members.

Member	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(10)**	**(11)**	**(12)**	**(13)**
$\bar{v}_i^x$	1	–0.5	0.5	–0.309	0.309	0	0.5	0.309	–0.809	0.309
$\bar{v}_i^y$	0	0.866	0.866	–0.178	–0.178	0.357	–0.646	0.756	–0.110	–0.178
$\bar{v}_i^z$	0	0	0	0.934	0.934	0.934	0.577	0.577	0.577	0.934

	(14)	**(15)**	**(16)**	**(17)**	**(18)**	**(22)**	**(23)**	**(24)**	**(25)**	**(26)**
	0	–0.309	0.5	0.5	1	0.309	–0.809	0.309	0	–0.809
	0.357	–0.178	–0.866	0.866	0	0.756	–0.110	–0.178	0.934	–0.467
	0.934	0.934	0	0	0	0.577	0.577	0.934	-0.357	–0.357

	(27)	**(28)**	**(29)**	**(30)**
	0.809	0.809	0	–0.809
	–0.467	–0.467	0.934	–0.467
	–0.357	–0.357	–0.357	–0.357

Table 3. Force vectors at self-equilibrium state.

Member	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**	**(10)**
v_i^x	0.982	–0.375	0.607	–0.202	0.327	–0.125	0.750	1.214	–1.963	0.577
v_i^y	–0.134	0.917	0.783	–0.261	–0.045	0.306	–1.834	1.567	0.268	–0.656
v_i^z	0	0	0	0.934	0.934	0.934	–1.401	–1.401	–1.401	0.467

	(11)	**(12)**	**(13)**	**(14)**	**(15)**	**(16)**	**(17)**	**(18)**	**(19)**	**(20)**
	0.280	–0.857	0.327	–0.125	–0.202	0.375	0.607	0.982	0.750	1.214
	0.828	–0.172	–0.045	0.306	–0.261	–0.917	0.783	–0.134	–1.834	1.567
	0.467	0.467	0.934	0.934	0.934	0	0	0	–1.401	–1.401

	(21)	**(22)**	**(23)**	**(24)**	**(25)**	**(26)**	**(27)**	**(28)**	**(29)**	**(30)**
	–1.964	0.280	–0.857	0.577	–0.048	–0.732	0.780	0.780	–0.048	–0.732
	0.268	0.828	–0.172	–0.656	0.873	–0.478	–0.395	–0.395	0.873	–0.478
	–1.401	0.467	0.467	0.467	–0.467	–0.467	–0.467	–0.467	–0.467	–0.467

4. Strongest Structure with Given Configuration

Many tensegrity structures have more than one independent mode of member forces, such that distribution of member forces can be controlled in some extents as long as the self-equilibrium conditions are satisfied. Since stiffness of a tensegrity structure is greatly influenced by its member forces, this also gives us the chance to achieve higher stiffness for the structure. This section is to present a bi-objective optimization problem for finding the optimal distribution of member forces, which leads to the strongest structure (with maximum stiffness), and have closely uniform member forces as well. Note that we assume that configuration of the structure is given, besides those assumptions listed at the end of Section 1.

4.1 Self-equilibrium and Stiffness

Let $\mathbf{s} \in \Re^{m}$ denote the vector of member forces. Self-equilibrium equation of the structure can be written as follows[5]

$$\mathbf{Ds} = \mathbf{0} \tag{15}$$

where the trivial vector $\mathbf{0}$ on the right-hand side of the equation indicates that there is no external load applied to the structure, and the matrix $\mathbf{D} \in \Re^{dn \times m}$ is determined by the geometry of the structure, i.e., the connectivity and the nodal coordinates. Hence, $\mathbf{D}$ is a constant matrix from the assumption that topology and configuration of the structure are known a priori.

Let R denote the rank of $\mathbf{D}$. If $R<m$, then there are $m-R$ independent modes $\mathbf{f}_i$ of member forces satisfying the self-equilibrium equation: $\mathbf{Df}_i = \mathbf{0}$. Member forces of the structure can be written as a linear combination of these modes through the coefficients α_i :

$$\mathbf{s} = \textstyle\sum_{i=1}^{m-R} \alpha_i \mathbf{f}_i = \mathbf{F}\boldsymbol{\alpha} \tag{16}$$

The tangent stiffness matrix $\mathbf{K}$, second-order derivative of the total potential energy, can be written as sum of the linear stiffness matrix $\mathbf{K}^{E}$ and the geometrical stiffness matrix $\mathbf{K}^{G}$ as follows[5]

$$\mathbf{K} = \mathbf{K}^{E} + \mathbf{K}^{G} \tag{17}$$

where $\mathbf{K}^{E}$ is always positive semi-definite for the structures having positive axial stiffness; and the positive definiteness of $\mathbf{K}^{G}$ depends on the distribution of member forces.

A non-trivial displacement is called *mechanism* if it does not change the member lengths. Let $\mathbf{M}$ denote the mechanism matrix for which the ith column is the ith independent mechanism. The quadratic form $\mathbf{Q}$ of $\mathbf{K}$ with respect to $\mathbf{M}$ turns out to be that of $\mathbf{K}^{G}$ as follows, because $\mathbf{M}^{\mathrm{T}}\mathbf{K}^{E}\mathbf{M} = \mathbf{O}$ holds[6, 7]

$$\mathbf{Q} = \mathbf{M}^{\mathrm{T}}\mathbf{K}\mathbf{M} = \mathbf{M}^{\mathrm{T}}\mathbf{K}^{G}\mathbf{M} \tag{18}$$

$\mathbf{Q}$ is positive definite if the structure is stable when no term higher than second-order terms of the total potential energy are considered; and it is sufficient to consider the positive definiteness of $\mathbf{Q}$ in the stability investigation of tensegrity structures, if the forces are small enough compared to the member stiffness[6]. Since $\mathbf{M}$ is constant when the geometry of the structure is determined, stability and stiffness of the structure is directly related to the distribution of member forces.

4.2 Maximum Stiffness and Uniform Member Forces

In this subsection, we formulate an optimization problem with two objectives: (a) maximization of the stiffness, and (b) minimization of the deviation of the member forces from the target values. Moreover, the signs of the member forces, i.e., tension for cables and compression for struts, are also incorporated as constraints.

Maximum Stiffness

From the stability criterion based on (18), the stability and stiffness of tensegrity structures can be defined by the smallest eigenvalue λ of $\mathbf{Q}$, when the member stiffness is large enough compared to the level of member forces. This is because that λ is also the smallest eigenvalue of $\mathbf{K}$ in this case. For simplicity, we assume that all members have infinite stiffness so that stability of the structure can be verified by the sign of λ : when λ is positive, the structure is stable; when it is negative, the structure is unstable. Since λ is the minimum eigenvalue of $\mathbf{K}$, or equivalently $\mathbf{Q}$, its corresponding eigenvector is the weakest direction for the structure to deform. Hence, to have a stronger structure, a distribution of member forces resulting in an increase of λ is to be found.

Uniform Member Forces

Uniform distribution of member forces has many advantages in design, construction and maintenance of tensegrity structures. For example, fabrication costs and complexity of construction process can be significantly reduced, if cross-sectional areas of the members are the same for the same type of them; moreover, these members have the same safety factor against member failure.

The target member forces are denoted by $\bar{\mathbf{s}}$. The difference $\|\mathbf{s}-\bar{\mathbf{s}}\|$ between $\mathbf{s}$ and $\bar{\mathbf{s}}$ is to be minimized as the other objective function. The least square method can simply give the optimal solution as follows if only this objective function is considered

$$\mathbf{s} = \mathbf{F}\mathbf{F}^{+}\bar{\mathbf{s}} \tag{19}$$

where $\mathbf{F}^{+}$ denotes the generalized inverse of $\mathbf{F}$.

Note that both of the two objectives mentioned above are described in terms of member forces. However, they cannot have global optimal solutions at the same time. A trade-off between them is generally required.

Constraints

The strain energy Π of the structure can be written as

$$\Pi = \sum s_i^2 l_i / 2A_i E_i \tag{20}$$

For simplicity, we assume that all members have the same $l_i / 2A_i E_i$, which is denoted as a. Hence, we have

$$\Pi = a\sum s_i^2 = a\mathbf{s}^{\mathrm{T}}\mathbf{s} \tag{21}$$

It is always desirable to obtain the strongest structure within a limited energy to be introduced to the structure. For this purpose, $\bar{\Pi}$ is used as a specified value for external energy introduced to the structure, where the constant a is ignored

$$\mathbf{s}^{\mathrm{T}}\mathbf{s} = \bar{\Pi} \tag{22}$$

Moreover, it is also expected that the member forces conform to the types of the members; i.e. tension for cables and compression for struts. Let $\mathbf{s}^c$ and $\mathbf{s}^s$ denote the member forces of cables and struts. Then we will have $\mathbf{s}^c > 0$ and $\mathbf{s}^s < 0$ for the constraints on the signs of the member forces.

Formulation of Optimization Problem

Based on the discussions on objective functions and constraints, the multiobjective optimization problem for force design of a tensegrity structure with give configuration is formulated as

$$\begin{array}{lll} \text{(C)} & \text{Minimize} & -\lambda \text{ and } \|\mathbf{s}-\bar{\mathbf{s}}\| \\ & \text{Subject to} & \mathbf{s}^c \geq \mathbf{0} \\ & & \mathbf{s}^s \leq \mathbf{0} \\ & & \mathbf{s}^{\mathrm{T}}\mathbf{s} = \bar{\Pi} \end{array}$$

A multiobjective optimization problem may have compromise solutions, called *Pareto optimal solutions*, in which it is impossible to improve all of the objectives at the same time. There have been many methods developed for this kind of problems, among which we adopt the constraint approach to find the Pareto optimal solutions as candidates for the assistance of decision making.

4.3 Example

In this section, we consider the force design of a special tensegrity structure, called tensegrity grid, proposed by Motro [8]. The tensegrity grid in Figure. 2 consists of 38 nodes and 115 members, is used an example structure. Height of the structure is 5.0, and the projection of each strut on *xy*-plane has length of 5.0 as well. There are eight independent modes of member forces in total, and the coefficients are to be determined for the presentation of member forces as in (16) by solving the problem (C).

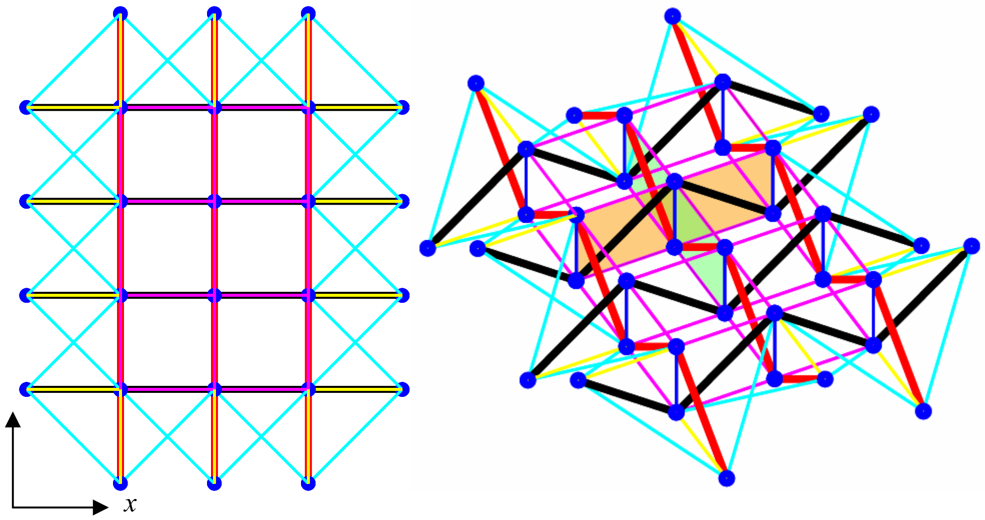
In order to find the Pareto optimal solutions, the constraint approach is adopted: the second objective function $\|\mathbf{s}-\bar{\mathbf{s}}\|$ of the problem (C) is incorporated into the constraints so that the original multiobjective problem is transformed into a single-objective problem as

$$\begin{array}{lll} \text{(D)} & \text{Minimize} & -\lambda \\ & \text{Subject to:} & \|\mathbf{s}-\bar{\mathbf{s}}\| \leq \varepsilon \\ & & \mathbf{s}^c \geq \mathbf{0} \\ & & \mathbf{s}^s \leq \mathbf{0} \\ & & \mathbf{s}^{\mathrm{T}}\mathbf{s} = \bar{\Pi} \end{array}$$

where ε is the upper bound of the difference between the member forces from their target values. The set of Pareto optimal solutions for the original bi-objective optimization problem (C) can be derived by solving the revised single-objective optimization problem (D), where ε for $\|\mathbf{s}-\bar{\mathbf{s}}\|$ is varied gradually and consecutively.

For the revised single-objective problem (D), we need the lower and upper bound of ε. The lower bound of ε can be determined by solving problem (C) ignoring the objective function $-\lambda$; it can also be easily found as the least square solution in (19). The upper bound of ε can be derived by solving the problem (C) to minimize $-\lambda$ only, ignoring the other objective function. The function *fmincon( )* in the Optimization Toolbox of MATLAB [9] is used to solve the single-objective problem (D). *fmincon( )* is a nonlinear programming routine, which attempts to find a constrained minimum of a scalar function with several variables, and starts from an initial estimate.

The target values for the member forces of struts and cables are set to –1 and 1, respectively. If the member forces exactly agree with the target values, the revised strain energy introduced to the structure is $\mathbf{s}^{\mathrm{T}}\mathbf{s}=101$ because there are 101 struts and cables in addition to 14 bars on the boundary carrying no forces. Hence, we set $\bar{\Pi}=101$ for the problem. Note that these values are purely numerical without explicit physical meaning. The coefficients α_i for the force modes $\mathbf{f}_i$ are the variables to be determined in the problem. The initial solution to start the *fmincon( )* is determined by the least square method as in (19).



(a) Top View (b) Bird View

Figure 4. An example of tensegrity grid.

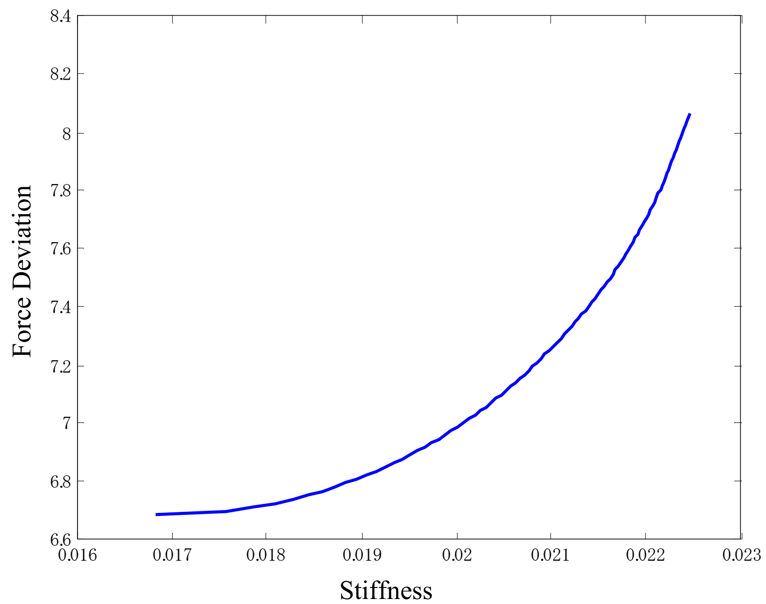


Figure 5. Pareto optimal solutions of problem (C) for maximum stiffness and uniform member forces.

The difference ε between the member forces and their specified target values is distributed in the region [6.6849, 8.0662]. ε is varied in this region to find its corresponding maximum λ by solving the problem (D). This way, we obtain the set of Pareto optimal solutions, which are plotted in Figure. 5. It can be observed from the figure that a trade-off relation between the two objective functions. Basically, larger difference between the member forces and the target forces leads to stronger structure, but they do not have linear relation. In the force design process, a compromise between the two objectives should be made. Curve of the Pareto optimal solutions can provide direct information to help designers to have deeper understanding of the structure and to assist their decision making in the force design.

5. Discussions and Conclusions

Three optimization approaches for shape design and force design have been discussed in this study. The first two approaches are for the determination of configurations and member forces at the state of self-equilibrium, by (1) minimizing difference between strain energy in cables and in struts, and (2) minimizing deviation of force components from their target values for the structures modeled as directed graphs; The third approach is to find the optimal distribution of member forces that leads to the strongest structure (with maximum stiffness) as well as minimum deviation of member forces from their target values, where configuration of the structure is assumed to be known.

The first proposed method by minimizing difference of strain energy between cables and struts can have precise control over magnitudes of some member forces. The difficulty in nondifferentiability at the boundary of the feasible region can be avoided by using the primal-dual interior-point methods. The problem for cable nets, which is composed of members carrying only tension but no compression, is convex, and hence, the primal-dual interior-point methods can efficiently find the global optimal solution for the determination of their configurations and member forces. However, it may fail in some cases of tensegrity structures since the problem for them is intrinsically non-convex.

In the second approach, the authors' method of directly assigning force components has been extended to the structures with moderately large numbers of nodes and members. We have the following discussions for the proposed method.

1. Member directions can be directly controlled by considering force components as variables, which is different from other approaches for the control of magnitudes of member forces or force-to-length ratios (force densities).
2. The force components are found as solutions of a convex quadratic programming problem with linear equality constraints, so that its solution can be easily obtained by solving linear equations derived from stationary conditions of the Lagrangian.
3. The configuration and forces of the tensegrity structures at the state of self-equilibrium can be directly controlled by modifying the target values and soft constraints on force components.

As discussed in the third approach, it is always desirable that the structure has stiffness as high as possible, in the design of a tensegrity structure subjected to given conditions. Moreover, nearly uniform distribution of member forces has many advantages, such as reduction of fabrication costs as well as complexity of construction process, and having the same safety factor for the failure of members.

For the structures having multiple force modes, it was shown in the study that we have the freedom to choose the member forces to influence mechanical properties of the structures. A multiobjective optimization problem was presented to maximize the stiffness and to minimize the difference between the member forces and their target values, subject to the constraints on given strain energy and types of members.

It is clear from the numerical example in Section 4 that distribution of member forces has significant influence on the stiffness of the structure, moreover, the highest stiffness and the nearly uniform distribution of member forces cannot be obtained at the same time. Presentation of the curve of the Pareto optimal solutions enables designers to select a solution from the candidate solutions according to their preferences.

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