Generation of Link Mechanism by Shape-Topology Optimization of Trusses Considering Geometrical Nonlinearity

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Abstract
A two-stage general optimization approach is presented for generating link mechanisms from a highly connected ground structure. The structure is modeled as a pin-jointed truss, and is to be optimized so that a large displacement is generated in the specified direction at the output node. The design variables are the cross-sectional areas of the members and the nodal locations. The equilibrium path of an unstable mechanism is traced by the displacement control method. In the first step, the unnecessary members are removed by solving the optimization problem for minimizing the total structural volume under constraints on the maximum load and the stiffnesses at the initial and final states. In the second step, the deviation of the displacement of the output node from the specified direction is minimized. It is shown in the numerical examples that several mechanisms can be naturally found as a result of the two-stage optimization starting from randomly selected initial solutions.

Keywords
mechanism, finite deformation, topology optimization, truss

1. Introduction

Many approaches have been presented for design of link mechanisms. However, most of the existing methods are based on trial-and-error modification of the design variables such as nodal locations (geometry) and member locations (topology).

Kawamoto [1] presented an approach based on the truss model, where the nodal locations are modified to realize the desired mechanism. Kawamoto *et al.* [2] developed an method based on graph enumeration. However, in these methods, the numbers of nodes and members should be assigned *a priori*. Kim *et al.* [3] used a model that consists of rigid bodies connected by springs, and the optimal topology is obtained by removing the unnecessary springs. However, in their method, the geometry is fixed and only a restricted design space is searched. Sekimoto and Noguchi [4] presented an optimization approach to trace the desired nodal path for finite-element models. However, they did not intend to generate a link mechanism. Recently, extensive researches are found for structural optimization under stability constraints [5, 6]. The authors developed an optimization method for generating multistable compliant bar–joint systems [7, 8].

In this study, the conditions for a truss to be a mechanism is first expressed by the rank and eigenvalues of the stiffness matrix. The method in [7, 8] for compliant bar–joint system is extended to generate link mechanisms that realize the specified path of the output node. A two-stage algorithm is presented based on the ground structure approach. The desired kinematic property is realized in the first stage, and the displacement of the output node perpendicular to the target direction is minimized in the second stage. Geometrically nonlinear path-following analysis is conducted at each step of optimization, and the cross-sectional areas and the locations of the nodes are optimized by a random-start mathematical programming approach [9].

2. Kinematical indeterminacy of the link mechanism

Consider a pin-jointed truss allowing elastic axial deformation of members. Suppose, for simplicity, that all the rigid-body motions are appropriately constrained. Let $\boldsymbol{P} \in \mathbb{R}^n$ and $\boldsymbol{N} \in \mathbb{R}^m$ denote the vectors of external loads and axial forces, where n and m are the degree-of-freedom of displacements and the number of members, respectively. The equilibrium equation is written as

$$\boldsymbol{DN} = \boldsymbol{P} \tag{1}$$

where $\boldsymbol{D} \in \mathbb{R}^{n\times m}$ is called equilibrium matrix. If $r = \text{rank}\ \boldsymbol{D} = m$, then the truss is statically determinate, and the axial forces $\boldsymbol{N}$ for a given $\boldsymbol{P}$ are found by solving the equilibrium equation (1). If $r < n$, then the truss is unstable, and there exists $h = n - r$ independent mechanisms, where h is called kinematical indeterminacy.

One of the following two approaches can be used for modeling and optimization of link mechanisms:

1. The members are assumed to be rigid, and the equilibrium path is traced by assigning the constraints such that the length of each member does not change. The optimal shapes to minimize the error of the path of the output node from the target path can be found by considering the nodal locations as design variables.

2. Elastic deformation of members is allowed, and the equilibrium path is traced by a conventional displacement control or arc-length approach of large deformation analysis of trusses. Unnecessary members are removed by optimization under constraint such that the maximum load vanishes throughout the equilibrium path to arrive at an unstable link mechanism.

In the first approach, the numbers of members and nodes should be appropriately selected and fixed during optimization so that the kinematical indeterminacy is equal to 1. Furthermore, many geometrical constraints are needed to keep the member length constant. Therefore, this approach is not desired for optimization that demands many analyses, and we use the second approach.

Let L_i and A_i denote the length and cross-sectional area of member i. Define $\boldsymbol{C} \in \mathbb{R}^{m\times m}$ as a diagonal matrix of which the (i, i) component is equal to $A_i E/L_i$, where E is Young's modulus. The stiffness matrix $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ is decomposed as

$$\boldsymbol{K} = \boldsymbol{D}\boldsymbol{C}\boldsymbol{D}^\top \tag{2}$$

Since $\boldsymbol{C}$ is full-rank, the rank of $\boldsymbol{K}$ is equal to the rank r of $\boldsymbol{D}$.

For an kinematically indeterminate link mechanism with $h = 1$, there exists a mode (mechanism) $\boldsymbol{\Phi}$ that satisfies $\boldsymbol{K}\boldsymbol{\Phi} = \boldsymbol{0}$. Then $\lambda_1 = 0$ and $\lambda_i > 0$ are satisfied for the eigenvalues λ_i $(i = 1, \dots, n)$ of the stiffness matrix $\boldsymbol{K}$.

An unstable structure with $h = 1$ can be stabilized by constraining the displacement corresponding to one nonzero component of $\boldsymbol{\Phi}$. Therefore, the equilibrium path can be traced by using this displacement component as control parameter.

3. Optimization problem for generating link mechanisms

Link mechanisms are generated by removing unnecessary members by optimization from a highly connected ground structure. We consider a plane structure for simplicity.

A displacement U_{a} (> 0) is given as shown in Fig. 1 at an input node 'a'. The final state is defined such that the displacement U_{b} of the output node 'b' reaches the specified value $\bar{U}_{\mathrm{b}}$ as

$$U_{\mathrm{b}} = \bar{U}_{\mathrm{b}} \tag{3}$$

The solid and dotted lines in Fig. 1 are the configurations before and after deformation, respectively. The truss is stable in the early stage of optimization, where the truss has many members, and the maximum value of the input load P_{a} before reaching the final state is not zero. In order to naturally generate an unstable mechanism, a constraint is given such that the maximum value $P_{\mathrm{a}}^{\max}$ of P_{a} applied in the specified direction at node 'a' vanishes.

Fig. 2 illustrates the load-displacement relation at the input node from the initial state to the final state. Since the maximum load cannot be negative, as illustrated in Fig. 2, the constraint on the maximum load is relaxed as

$$P_{\mathrm{a}}^{\max} \leq 0 \tag{4}$$

It is confirmed in the following examples that the constraint (4) is satisfied in equality at the optimal solution. In the optimization algorithm, a small tolerance ε is usually assigned for constraints. Therefore, (4) is equivalent to

$$P_{\mathrm{a}}^{\max} \leq \varepsilon \tag{5}$$

The design variables are the vectors of cross-sectional areas $\boldsymbol{A}$ and the nodal coordinates $\boldsymbol{X}$. In order to obtain a mechanism with a small number of members, the total structural volume $V(\boldsymbol{A}, \boldsymbol{X})$ is minimized as objective function. If

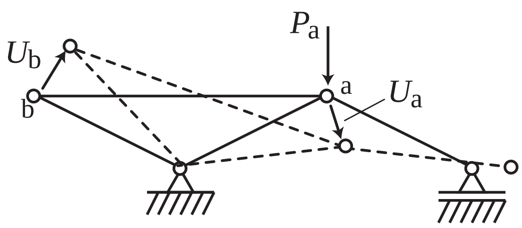


Figure 1 A link mechanism.

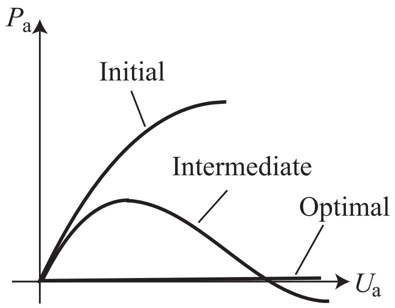


Figure 2 Relation between input force P_a and input displacement U_a of initial, intermediate and optimal solutions.

we assign the constraint (4) only, then all the cross-sectional areas vanish as a result of minimizing the total structural volume. Therefore, the stiffness constraints are given in terms of nodal displacements as

$$U_{bx}^{f0} \leq \bar{U}_{bx}^{f0}, \tag{6a}$$
$$U_{by}^{f0} \leq \bar{U}_{by}^{f0} \tag{6b}$$

where U_{bx}^{f0} and U_{by}^{f0} are the displacements of the output node 'b' in x- and y-directions against unit loads in x- and y-directions at node 'b', respectively, after constraining the input degree of freedom, and $\bar{U}_{bx}^{f0}$ and $\bar{U}_{by}^{f0}$ are their specified upper bounds. Note that the tangent stiffness at the deformed configuration is used for evaluating U_{bx}^{f0} and U_{by}^{f0} [7, 8].

Hence, the optimization problem for finding the link mechanism is stated as

$$\text{minimize} \quad V(\boldsymbol{A}, \boldsymbol{X}) \tag{7a}$$
$$\text{subject to} \quad P_a^{\max}(\boldsymbol{A}, \boldsymbol{X}) \leq 0, \tag{7b}$$
$$U_{bx}^{f0}(\boldsymbol{A}, \boldsymbol{X}) \leq \bar{U}_{bx}^{f0}, \tag{7c}$$
$$U_{by}^{f0}(\boldsymbol{A}, \boldsymbol{X}) \leq \bar{U}_{by}^{f0}, \tag{7d}$$
$$\boldsymbol{A}^L \leq \boldsymbol{A}, \tag{7e}$$
$$\boldsymbol{X}^L \leq \boldsymbol{X} \leq \boldsymbol{X}^U \tag{7f}$$

where $\boldsymbol{X}^U$ and $\boldsymbol{X}^L$ are the upper and lower bounds of $\boldsymbol{X}$, and $\boldsymbol{A}^L$ is a small lower bound for $\boldsymbol{A}$. Note that the member with $A_i = A_i^L$ at the optimal solution is to be removed.

The optimization algorithm is given as

Step 1 Assign initial values for $\boldsymbol{A}$ and $\boldsymbol{X}$.

Step 2 Trace the equilibrium path by using the input displacement at node 'a' as the control parameter until reaching the final state satisfying the condition (3).

Step 3 Compute the objective and constraint functions and their sensitivity coefficients.

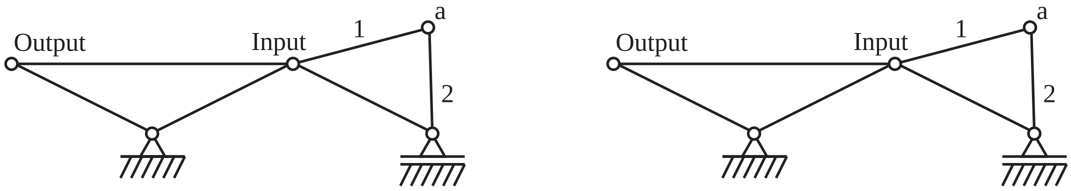
Step 4 Update the variables according to the optimization algorithm.

Step 5 Go to Step 2 if the convergence criteria are not satisfied.

In Step 2, the optimization process is terminated if the output displacement is not in the specified direction, because the desired mechanism is not likely to be obtained. Furthermore, the optimization problem is highly nonlinear, and many local optimal solutions are expected to exist. Therefore, the initial values of $\boldsymbol{A}$ and $\boldsymbol{X}$ are randomly generated, and the path-tracing is continued only for the case where the output displacement U_b is in the specified direction.

The optimal solution of this first-stage problem is unstable and satisfies the requirements for link mechanism. However, the displacement $\tilde{U}_b$ of the output node in the direction perpendicular to the specified direction may be large, because no constraint has been given on the direction of the output displacement. Therefore, in the second-stage problem, the maximum absolute value $\tilde{U}_b^{\max}$ of $\tilde{U}_b$ is minimized by considering $\boldsymbol{X}$ as design variables, while $\boldsymbol{A}$ is fixed, as

$$\text{minimize} \quad \tilde{U}_b^{\max}(\boldsymbol{X}) \tag{8a}$$
$$\text{subject to} \quad \boldsymbol{X}^L \leq \boldsymbol{X} \leq \boldsymbol{X}^U \tag{8b}$$



(a) Case 1 (a) Case 2

Figure 3 Unnecessary members.

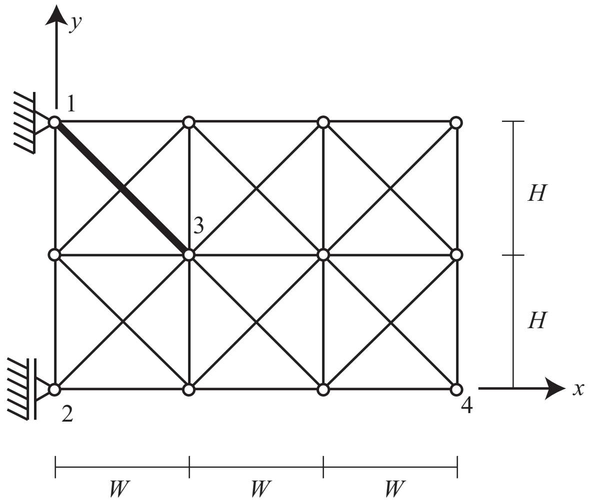


Figure 4 A plane truss model.

Hence, by solving the two-stage problems, the link mechanisms with desired property can be obtained.

Note again that the unnecessary members are removed in the first-stage problem for minimizing the total structural volume. However, since the problem is highly nonlinear, strict convergence to the global optimal solution may not be expected. Even for the case where global optimum is not found, a feasible solution is obtained in the first stage, and we can proceed to the second stage. Since the topology is fixed at the second stage, unnecessary members cannot be removed at this stage. Note that the purpose of this study is not to obtain the global optimal solution, but to obtain a mechanism that has the desired kinecatic property.

Therefore, the unnecessary members are manually removed after completing the second stage. If a member is removed without changing the load-displacement relation, then the member is considered to be unnecessary. Such a member is detected as follows:

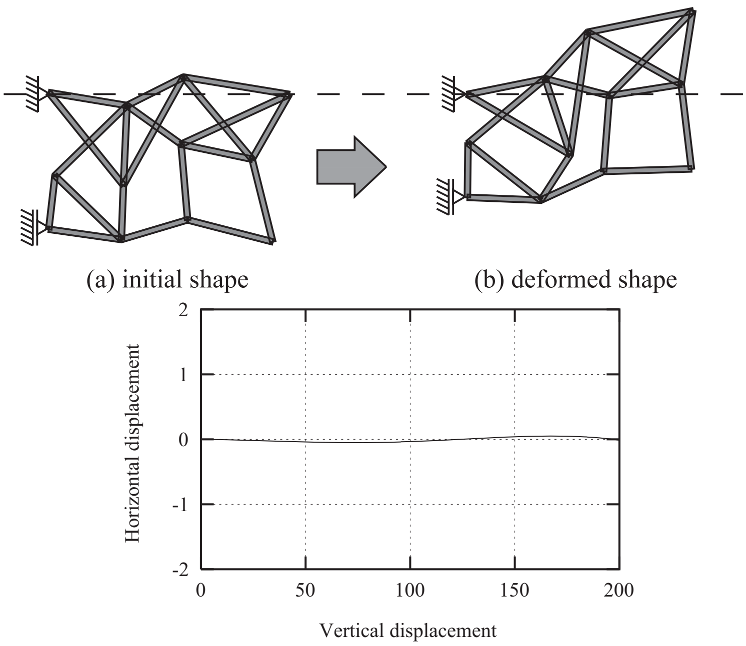
1. If no stress exists in the members against the input force after constraining the output displacement, then the member is considered to be unnecessary. For example, if only two members are connected to the nodes except the input node, output node, and supports, these members are unnecessary. The two members connected to node ‘a’ in Fig. 3(a) can be removed.

2. If a statically indeterminate substructure exists, and the global property does not change after removing a member in the substructure, then the member is unnecessary. For example, the quadratic substructure consisting of nodes ‘a–d’ is statically indeterminate, and members 1 or 2 can be removed without losing the stability of the substructure.

In the following numerical examples, the performance of the structure after removing the members are confirmed by path-tracing analysis.

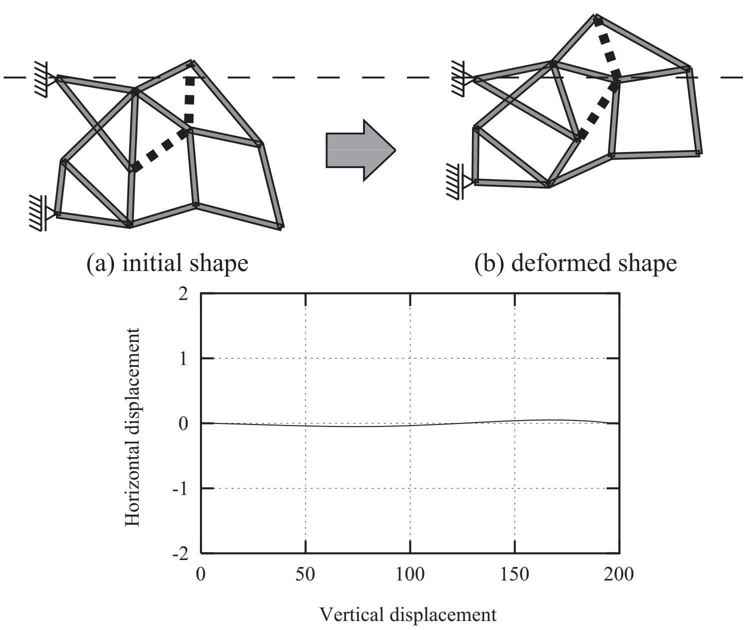
4. Numerical examples

Link mechanisms are found from the ground structure as shown in Fig. 4, where $W = H = 200$ mm. The truss is pin-supported at node 1, and fixed in x-direction at node 2. The ground structure has a 3×2 rectangular grid, and the intersecting diagonal members are not connected with each other.



(c) Relation between vertical and horizontal displacements of node 4.

Figure 5 Optimal solution 1.

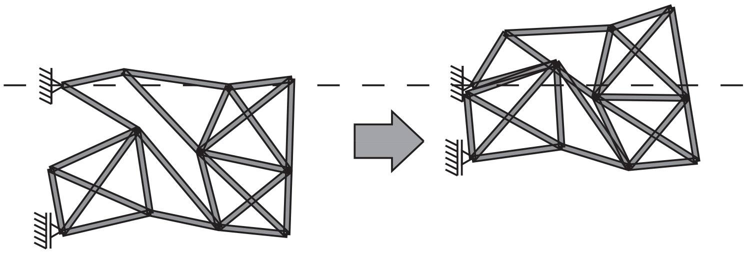


(c) Relation between vertical and horizontal displacements of node 4.

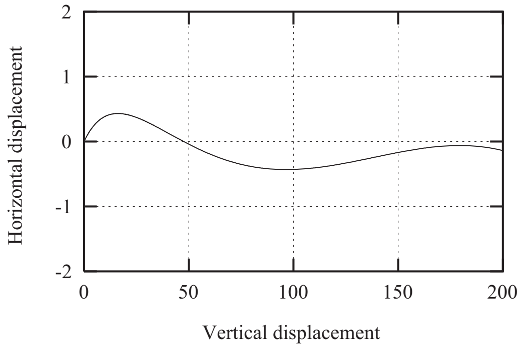
Figure 6 Optimal solution 1 after removing unnecessary members.

We obtain the mechanism so that node 4 moves 200 mm in y-direction as the result of anticlockwise rotation of node 3 around support 1. For this purpose, a large value is given for the lower bound A_i^{L} of the cross-sectional area of the thick member in Fig. 4 connecting support 1 and node 3. Since the optimal solution is unstable, the value of A_i^{L} does not have any effect on the performance of the link mechanism obtained by optimization. The anticlockwise rotation is realized by the forced y-directional displacement at node 3. Hence, the final state is defined as the deformation where the output displacement U_{b} in y-direction of node 4 reaches 200 mm as the result of y-directional input displacement U_{a} at node 3. The unit of force is N, and the upper bound displacements against unit loads in the first stage is given as $\bar{U}_{\mathrm{b}x}^{\mathrm{f0}} = \bar{U}_{\mathrm{b}y}^{\mathrm{f0}} = 50$ mm. The displacement in x-direction of node 4 is minimized in the second-stage of the optimization.

The cross-sectional areas of members except the thick member connecting support 1 and node 3 are considered as

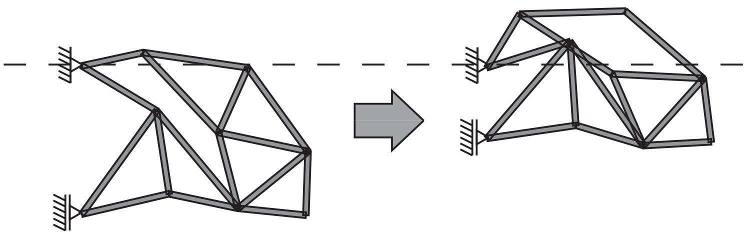


(a) initial shape (b) deformed shape

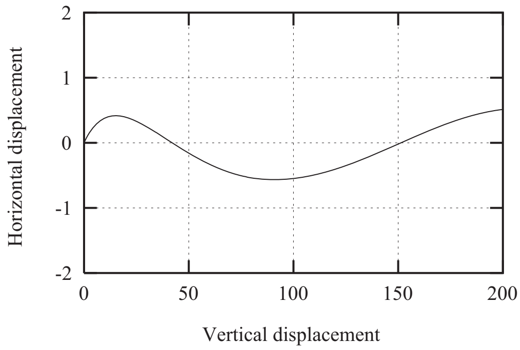


(c) Relation between vertical and horizontal displacements of node 4.

Figure 7 Optimal solution 2.



(a) initial shape (b) deformed shape

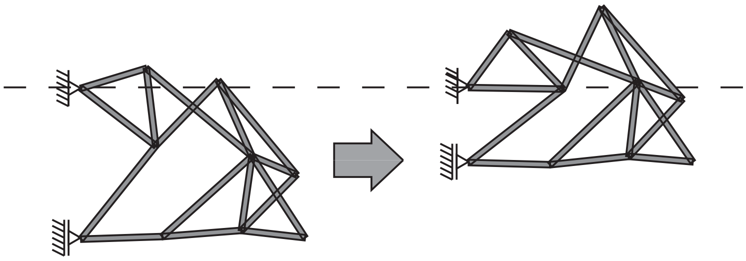


(c) Relation between vertical and horizontal displacements of node 4.

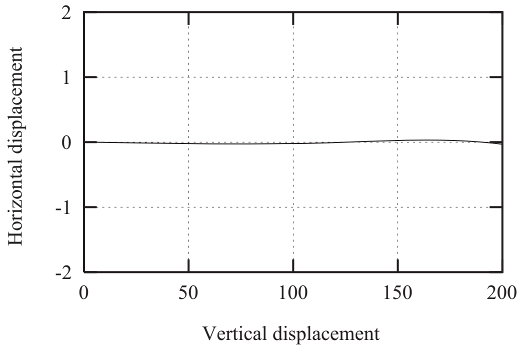
Figure 8 Optimal solution 2 after removing unnecessary members.

independent variables for which the lower bounds are equal to 0.01 mm^2. The x-coordinate of node 3, the x, y-coordinates of node 4, and the coordinates in the constrained directions of the supports are fixed, and the remaining coordinates are the independent design variables. In order to prevent the distortion of the topology, the bounds in the constraint (7f) for the nodal coordinates are given as the the rectangular region bounded by ±60 mm from the original location defined in Fig. 4.

The rotated engineering strain is used, and Young's modulus E is 2.0 N/mm^2. Since the final mechanism is unstable, the value of E does not affect the optimal solution. The y-directional displacement of node 3 is chosen as the parameter for the path-tracing analysis, where the increment of displacement is 1.0 mm, and the unbalanced loads are canceled at the subsequent step without carrying out iterative correction. In the following, the unit of length and displacement is mm.

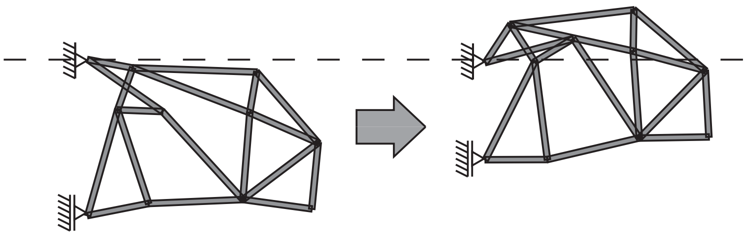


(a) initial shape (b) deformed shape

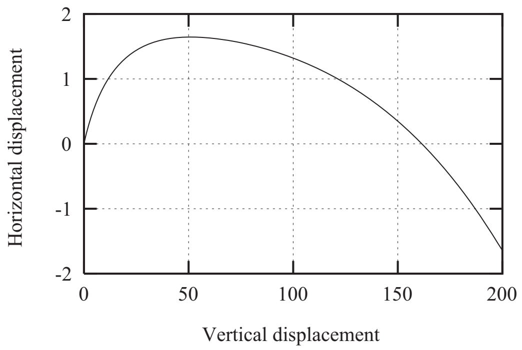


(c) Relation between vertical and horizontal displacements of node 4.

Figure 9 Optimal solution 3 after removing unnecessary members.



(a) initial shape (b) deformed shape



(c) Relation between vertical and horizontal displacements of node 4.

Figure 10 Optimal solution 4 after removing unnecessary members.

The performance of the mechanism is confirmed by ADAMS 2005 [10]. The sequential quadratic programming program in IDESIGN Ver. 3.5 [11] is used for optimization, where the tolerance for the constraints is $\varepsilon = 1.0 \times 10^{-4}$. The sensitivity coefficients are evaluated by finite difference approach, because numerical efficiency is not important in this study.

In order to obtain various mechanisms, initial solutions are generated randomly. Since the ratios rather than the absolute values are important for the cross-sectional areas, the initial values are defined so as to distribute uniformly between 0 and an appropriate positive value A_0 as

$$A_i = A_0 + 2A_0(R_i - 0.5) \tag{9}$$

where $R_i \in [0,1)$ is a uniform random value, and $A_0 = 10$ in the following examples. The initial values for the nodal coordinate x_i of the ith node is given as

$$x_i = x_i^0 + 120.0(R_i - 0.5) \tag{10}$$

where x_i^0 is the coordinate in Fig. 4. The y-coordinates are defined in a similar manner.

A mechanism obtained by the proposed two-stage optimization algorithm is shown in Fig. 5(a). Fig. 5(b) shows the deformed final shape, where the dashed horizontal line is added to clearly indicate that the y-coordinate is fixed at support 1. Fig. 5(c) shows the relation between y-directional and x-directional displacements of the output node 4. Note that horizontal and vertical axes of the figure correspond to vertical (y-directional) and horizontal (x-directional) displacements, respectively, and their scales are different. As is seen, node 4 moves straight in the y-direction.

The shape after removing the unnecessary members is shown in Fig. 6, where the figures (a), (b) and (c) are the initial shape, final shape, and the relation between y-directional and x-directional displacements of node 4. The dotted members have been added to obtain simpler configuration. Note that the members that can be added have been assumed to be selected from the ground structure. However, simpler configuration can be obtained if members can be added between any pair of nodes. The numbers of members and degree-of-freedom are 18 and 19, respectively, after removing the unnecessary members. Therefore, the kinematical indeterminacy h is equal to 1, if the equilibrium matrix is full-rank.

Another mechanism obtained from different initial solution is shown in Fig. 7, and the results after removing unnecessary members is shown in Fig. 8. Two different mechanisms are also shown in Figs. 9 and 10, where only the configurations after removing the unnecessary members are presented. Although the horizontal displacement of node 4 is rather large in solutions 2 and 4, it is very small in solution 3. The number of members after removing the unnecessary members are 16, 16 and 18 for solutions 2, 3 and 4, respectively, and the numbers of degree-of-freedom are 17, 17 and 19. Therefore, also for these mechanisms, the kinematical indeterminacy is equal to 1, if the equilibrium matrix is full-rank.

5. Conclusions

The following conclusions have been obtained from this study.

1. Link mechanisms can be found by two-stage truss optimization considering geometrical nonlinearity. In the first stage, the total structural volume is minimized under constraint on maximum load and the nodal displacements representing the stiffness of the truss, where the cross-sectional areas and nodal locations are considered as design variables. In the second stage, the nodal locations are optimized to obtain a mechanism for which the output node moves in the desired direction.

2. The equilibrium path can be traced by displacement control approach, if the degree of kinematical indeterminacy is 1. Therefore, optimal mechanisms can be found easily by utilizing the conventional large-deformation finite-element analysis program.

3. Several mechanism to convert input rotation to output translation have been successfully found by the proposed method.

4. In the proposed approach, the stiffness constraints are given only at the final state. The method to ensure stiffness throughout the equilibrium path need to be developed.

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