SYMMETRIC PRISMATIC TENSEGRITY STRUCTURES

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ABSTRACT

This paper presents a simple and efficient method to determine the self-equilibrated configurations of prismatic tensegrity structures with dihedral symmetry. It is demonstrated that stability of prismatic tensegrity structures is not only determined by the connectivity manner of the members, but also sensitive to the height/radius ratio and the stiffnss/prestress ratio. A catalogue of symmetric prismatic tensegrity structures with relatively small number of members is presented based on the stability investigations.

1. INTRODUCTION

In this paper, we describe a study into the configuration and stability of prismatic tensegrity structures with dihedral symmetry. The simplest example of this class of structures is shown in Figure. 1. This class of structures was studied by Connelly and Terrell [1]: they showed that the example shown in Figure. 1, and other prismatic tensegrity structures where the horizontal cables are connected to adjacent nodes, are guaranteed to be stable, regardless of the levels of prestress and material properties. These structures are called *super stable*. However, the stability of other structures in this class was not addressed, and is the subject of this paper.

We show that, in general, the stability of prismatic tensegrity structures depends not only on the connectivity of the members, but also on their geometry (height/radius ratio), and also on the level of prestress and the stiffness of struts and cables.

2. SYMMETRY AND CONFIGURATION

We are considering structures that have dihedral symmetry, denoted $\mathbf{D}_n$: there is a single major n-fold rotation axis, which we assume is the vertical, z-axis, and n 2-fold rotation axes perpendicular to this. The structures consist of $2n$ nodes, arranged in two horizontal planes, with n nodes in each. We number the nodes from 0 to $n-1$ in the top plane, and n to $2n-1$ in the bottom plane.

Each node of the structure is connected by two horizontal cables within its own horizontal plane, and is connected by one 'vertical' cable and one strut to nodes in the other plane. An example

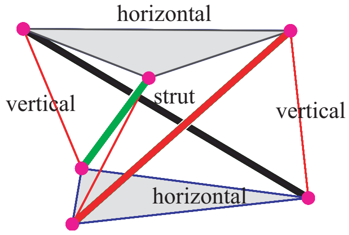


Fig. 1: The simplest prismatic tensegrity structure. The thick and thin lines denote, respectively, cables that can only carry tension, and struts that carry compression. There are two horizontal planes, which have here been coloured grey to aid perception. This structure has $\mathbf{D}_3$ symmetry, and is denoted $\mathbf{D}_3^{1,1}$.

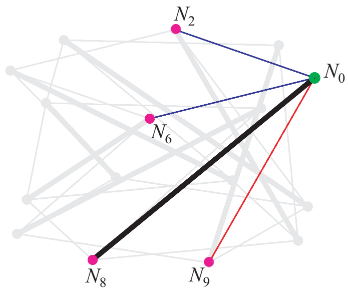


Fig. 2: All nodes connected to node 0, N_0, of an example tensegrity structure with $\mathbf{D}_8$ symmetry. The horizontal cables are connected to nodes 2 and $n-2=6$, the strut is connected to node $n=8$, and the 'vertical' cable is connected to node $n+1=9$. This structure is denoted $\mathbf{D}_8^{2,1}$.

showing the cables and struts connected to node 0, N_0, is shown in Figure. 2. Node N_0 is connected by a strut to node N_n, by horizontal cables to nodes N_h and N_{n-h} (we assume that $h \le n/2$) and by a vertical cable to node N_{n+v}, where h and v are parameters that define the structure. We denote the structure defined by n, h and v as $\mathbf{D}_n^{h,v}$.

Each node is transformed into exactly one other node by one of the symmetry operations in the group (the nodes form a *regular orbit*). Each symmetry operation can be represented by a transformation matrix $\mathbf{R}_i$, where the operation transforms node N_0 to node N_i. Let the coordinates of nodes N_0 and N_i be denoted by $\mathbf{x}_0$ and $\mathbf{x}_i$ ($\in \Re^3$) in three-dimensional space, respectively. They are related by $\mathbf{R}_i \in \Re^{3\times 3}$ as

$$\mathbf{x}_i = \mathbf{R}_i \mathbf{x}_0 \tag{1}$$

where i runs from 0 to $2n-1$. The matrices $\mathbf{R}_i$ are given in Table 1. In group representation theory, the mapping from the symmetry operations to these matrices is said to form a *reducible representation* for the group [2].

To find a totally symmetric state of self-stress in the structure, we only have to consider equilibrium of one node — every other node is symmetrically equivalent [3]. We will consider node N_0. Let q_h, q_s and q_v denote the force densities — axial force f_i to length l_i ratio; i.e., $q_i = f_i/l_i$, of the horizontal cables, strut and vertical cable, respectively.

The two nodes connecting to N_0 as horizontal cables must be chosen as a pair; i.e., if N_h is selected to connect with N_0, then symmetry requires that node N_{n-h} should also be chosen. The coordinates $\mathbf{x}_h$ and $\mathbf{x}_{n-h}$ can be computed as follows

$$\mathbf{x}_h = \mathbf{R}_h \mathbf{x}_0, \quad \mathbf{x}_{n-h} = \mathbf{R}_{n-h}\mathbf{x}_0 \tag{2}$$

and the axial force vectors $\mathbf{f}_h$ and $\mathbf{f}_{n-h}$ of horizontal cables can be written as

$$\begin{aligned} \mathbf{f}_h &= f_h(\mathbf{x}_h - \mathbf{x}_0)/l_h = q_h(\mathbf{R}_h - \mathbf{I}^3)\mathbf{x}_0 \\ \mathbf{f}_{n-h} &= f_h(\mathbf{x}_{n-h} - \mathbf{x}_0)/l_h = q_h(\mathbf{R}_{n-h} - \mathbf{I}^3)\mathbf{x}_0 \end{aligned} \tag{3}$$

Table 1: Transformation matrices of the dihedral group $\mathbf{D}_n$. Note that each matrix has a block-diagonal form, where the non-zero entries occur in a 2×2 block in the top-left and a 1×1 block in the bottom right. The mapping from the operations to these submatrices forms an *irreducible* representation of the group [2]. We use the notation $C_i = \cos(2i\pi/n)$ and $S_i = \sin(2i\pi/n)$.

	$0 \le i \le n-1$	$n \le i \le 2n-1$
$\mathbf{R}_i$	$\begin{bmatrix} C_i & -S_i & 0 \\ S_i & C_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_i & S_i & 0 \\ S_i & -C_i & 0 \\ 0 & 0 & -1 \end{bmatrix}$

where $\mathbf{I}^3$ denotes the 3-by-3 identity matrix.

Unlike the horizontal cables, there is only one node N_{n+j} in the lower plane connected to N_0 as a strut or vertical cable — the inverse of $\mathbf{R}_{n+j}$ $(0 \leq j < n)$ is identical to itself; i.e., $\mathbf{R}_{n+j}^{-1} = \mathbf{R}_{n+j}$. The axial force vectors $\mathbf{f}_s$ and $\mathbf{f}_v$ of the strut and vertical cable are

$$\mathbf{f}_s = q_s(\mathbf{R}_{n+s} - \mathbf{I}^3)\mathbf{x}_0, \quad \mathbf{f}_v = q_v(\mathbf{R}_{n+v} - \mathbf{I}^3)\mathbf{x}_0 \tag{4}$$

We are interested in the case when the structure is in equilibrium without external loads. Thus, the node N_0 should be in a state of self-equilibrium:

$$\mathbf{f}_h + \mathbf{f}_{n-h} + 2\mathbf{f}_s + 2\mathbf{f}_v = \mathbf{0} \tag{5}$$

Substituting from (3), (4) and Table 1 gives

$$\mathbf{A}\mathbf{x}_0 = \mathbf{0} \tag{6}$$

where

$$\mathbf{A} = 2q_h \begin{bmatrix} C_h & 0 & 0 \\ 0 & C_h & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2q_s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 2q_v \begin{bmatrix} C_v & S_v & 0 \\ S_v & C_v & 0 \\ 0 & 0 & -1 \end{bmatrix} - 2(q_h + q_s + q_v) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Notice that $\mathbf{A}$ has a block-diagonal form, where the non-zero entries occur in a 2×2 block in the top-left and a 1×1 block in the bottom right

In order for (6) to give a solution for $\mathbf{x}_0$ that does not lie either in the xy-plane, or along the z-axis, then both the submatrices in $\mathbf{A}$ must be singular, and this gives the two following conditions

$$\begin{aligned} q_v &= -q_s \\ \frac{q_h}{q_v} &= \pm \frac{\sqrt{2 - 2C_v}}{1 - C_h} \end{aligned} \tag{8}$$

Since both of q_h and q_v should be positive for the cables, only the positive solution is adopted.

The general solution $\mathbf{x}_0$ of (6), the null-space of $\mathbf{A}$, is then given by

$$\mathbf{x}_0 = \frac{r}{r_0} \begin{bmatrix} C_v - 1 + \sqrt{2 - 2C_v} \\ S_v \\ 0 \end{bmatrix} + H/2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

where r_0 is the norm of the first vector representing the coordinates in xy-plane, and r and H are the radius and height of the structure, which can have arbitrary real values.

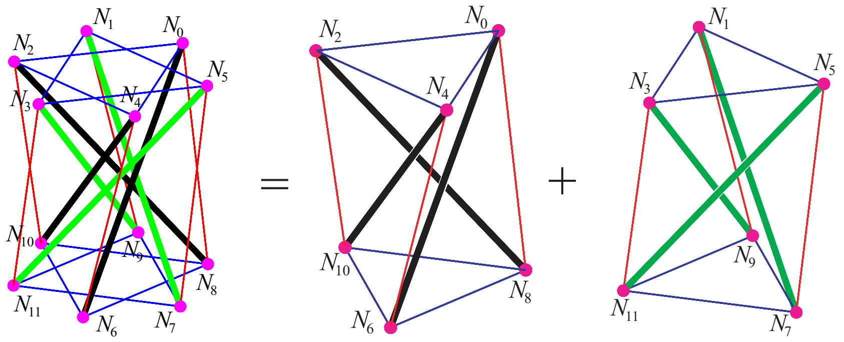
By the application of (1), then coordinates of all the nodes N_i of the structure can be determined by running i from 0 to $2n - 1$.

3. DIVISIBILITY CONDITIONS

Depending on the connectivity of members, a prismatic tensegrity structure may be completely separated into several identical substructures that have no mechanical relation with each other. For example, the structure $\mathbf{D}_6^{2,2}$ in Figure. 3 can be divided into two identical substructures $\mathbf{D}_3^{1,1}$.

We will exclude divisible structures from our stability investigation: the disconnected substructures have nothing to prevent relative motion. The substructures themselves can be considered as individual structures with lower symmetry.

This section presents the necessary and sufficient conditions for the divisibility of prismatic tensegrity structures.



(a) $\mathbf{D}_6^{2,2}$ (b) $\mathbf{D}_3^{1,1}$ (c) $\mathbf{D}_3^{1,1}$

Fig. 3: An example of divisible structure $\mathbf{D}_6^{2,2}$ and its two substructures $\mathbf{D}_3^{1,1}$.

3.1 Divisibility of Horizontal Cables

Suppose that we randomly select one node as the starting node, and travel to the next along the horizontal cables. If we repeat this in a consistent direction, eventually, we must come back to the starting node. The nodes and horizontal cables that have been visited in the trip are said to belong to the same *circuit*. If there are more than one circuit in the plane, the horizontal cables are said to be *divisible*; otherwise, they are *indivisible*.

Denote the number of circuits of the horizontal cables in one plane by n^c, and the number of nodes in each circuit by n^s. Each time we travel along a cable of one circuit, we pass by h nodes, and hence by the time we return to the original node, we have passed hn^s nodes. Suppose that in this circuit, we have travelled around the plane h^s times, and have hence passed by nh^s nodes. Thus,

$$n^s h = h^s n. \tag{10}$$

The number of circuits n^s in each horizontal plane is given by

$$n^c = \frac{n}{n^s} = \frac{h}{h^s}. \tag{11}$$

The necessary and sufficient condition for the divisibility of horizontal cables in the same plane is that there is more than one circuit of nodes, $n^c \neq 1$, and hence.

$$h \neq h^s \tag{12}$$

Valuable information about possible substructures can be found:

- The connectivity of the horizontal cables is h^s.
- The number of nodes in each plane is n^s.
- There are n^c substructures.

Note that (12) is only the divisibility condition for the horizontal cables, but not for the whole structure. Divisibility of vertical cables should also be taken into consideration.

3.2 Divisibility of Vertical Cables

If the horizontal nodes are divisible, then the nodes in the circuits of horizontal cables containing N_0 and N_n are

$$\begin{aligned} &\text{Circuit 1: } N_0, N_h, N_{2h}, \ldots, N_{(n^s-1)h} \\ &\text{Circuit 2: } N_n, N_{n+h}, \ldots, N_{n+(n^s-1)h} \end{aligned} \tag{13}$$

Circuit 1 and Circuit 2 are connected by struts. If they are also connected by cables, then the substructure constructed from these nodes can be completely separated from the original structure. Thus the structure will be divisible if the following holds, where v^s is an integer:

$$v = v^s h \tag{14}$$

In summary, (12) and (14) are the necessary and sufficient conditions for a divisible structure. The original structure $\mathbf{D}_n^{h,v}$ can be divided into n^c identical substructures $\mathbf{D}_{n^s}^{h^s,v^s}$.

4. STABILITY

In this section, the critical factors for the stability of prismatic tensegrity structures are investigated: height/radius ratio, connectivity, and stiffness/prestress ratio. We will use the symmetry-adapted coordinate systems to simplify our calculations, and present the results [4, 5].

4.1 Prestress Stability

If we assume that the axial stiffness of struts and cables is infinite, then to first order, the only way that the structure can deform is along the path of infinitesimal mechanisms. Then we can define prestress stability as follows:

If the quadratic form of the geometrical stiffness matrix with respect to the mechanisms is positive definite, then the structure is said to be prestress stable. [6]

This criterion is very convenient for the initial investigation of the stability of our structures, because the selection of materials does not need to be considered.

We will consider the calculation in a symmetry-adapted form, where we can separately consider the properties of the structure in different symmetry subspaces. Each symmetry subspace corresponds to one of the irreducible representations of the group. For the dihedral symmetry group $\mathbf{D}_n$, the irreducible representations are, for n even, $A_1, A_2, B_1, B_2, E_1, \cdots, E_{n/2-1}$, and for n odd, $A_1, A_2, E_1, \cdots, E_{(n-1)/2}$.

Let μ denote an irreducible representation of the symmetry group of the structure. The blocks of the symmetry-adapted geometrical stiffness matrix $\tilde{\mathbf{K}}_G$ and equilibrium matrix $\tilde{\mathbf{D}}$ corresponding to μ are denoted by $\tilde{\mathbf{K}}_G^\mu$ and $\tilde{\mathbf{D}}^\mu$, respectively. The symmetry-adapted mechanisms lying in the null-space of the transpose of $\tilde{\mathbf{D}}^\mu$ are written as columns of $\tilde{\mathbf{M}}^\mu$. Then, the block $\tilde{\mathbf{A}}^\mu$ corresponding to the representation μ of the symmetry-adapted quadratic form $\tilde{\mathbf{A}}$ of the geometrical stiffness matrix with respect to the mechanisms is

$$\tilde{\mathbf{A}}^\mu = (\tilde{\mathbf{M}}^\mu)^\top \tilde{\mathbf{K}}_G^\mu \tilde{\mathbf{M}}^\mu \tag{15}$$

The structure is prestress stable if and only if $\tilde{\mathbf{A}}^\mu$ are all positive definite for all representations μ except for A_2 and E_1, which corresponds to the rigid-body motions; and the positive definiteness of $\tilde{\mathbf{A}}^\mu$ can be easily verified because it is a matrix with dimensions of only one or two.

4.2 Critical Factors

Here, we show that the prestress stability of a prismatic tensegrity structure is not only influenced by the connectivity of horizontal cables but also that of the vertical cables, and furthermore, is sensitive to the height/radius ratio.

4.2.1 Height/Radius Ratio

Consider the structure $\mathbf{D}_8^{2,3}$ as an example. The structure is indivisible, and the relationship between the minimum eigenvalues of $\tilde{\mathbf{A}}^\mu$ and the height/radius ratio is plotted in Figure. 4.

The minimum eigenvalues of the $\tilde{\mathbf{A}}^{A_2}$ and $\tilde{\mathbf{A}}^{E_1}$ blocks are always equal to zero because they corresponds to the rigid-body motions. $\tilde{\mathbf{A}}^{A_1}$ and $\tilde{\mathbf{A}}^{E_2}$ are always positive definite, while the positive definiteness of $\tilde{\mathbf{A}}^{E_3}$ varies depending on the height/radius ratio.

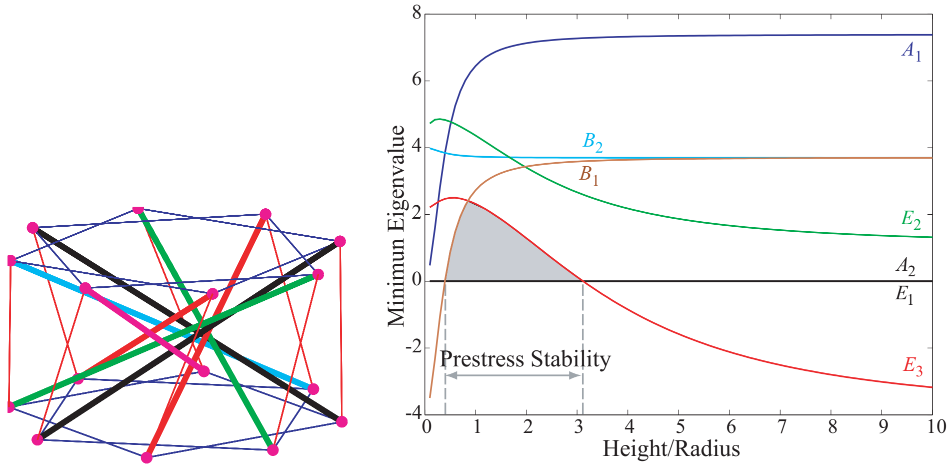


Fig. 4: The influence of the height/radius ratio on the prestress stability of the structure $\mathbf{D}_8^{2,3}$. The structure is prestress stable when the ratio is in the range $[0.4, 3.1]$

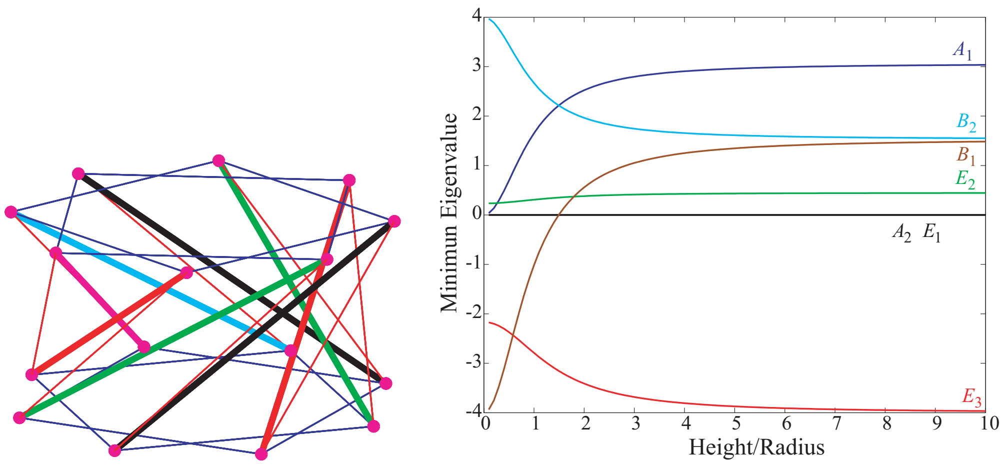


Fig. 5: The influence of the height/radius ratio on the prestress stability of the structure $\mathbf{D}_8^{2,1}$. The structure is never prestress stable.

We can see that the structure is prestress stable only when the height/radius ratio falls into the region $[0.4, 3.1]$, which is shown as a shaded area in the figure.

4.2.2 Connectivity

A structure is super stable only if $h = 1$ [1]. Thus it is clear that stability depends on the connectivity of horizontal cables. It has been illustrated above that in some special cases with the right height/radius ratio, the structure can still be prestress stable, even though it is not super stable. However, this is also dependent upon the connectivity of vertical cables.

For example, consider the structure $\mathbf{D}_8^{2,1}$ in Figure. 5, with the same connectivity of horizontal cables as $\mathbf{D}_8^{2,3}$, but different connectivity of vertical cables. Unlike the structure $\mathbf{D}_8^{2,3}$, the structure $\mathbf{D}_8^{2,1}$ can never be prestress stable because the minimum eigenvalue of $\tilde{\mathbf{A}}^{E_3}$ is always negative.

4.2.3 Materials and Self-stresses

This section will show the effect on the stability of the structures of having non-infinite stiffness for the cables and struts. We will make the simplification that all of the struts and cables have the same axial stiffness. The key parameter is then the ratio of the axial stiffness to the prestress in the structure.

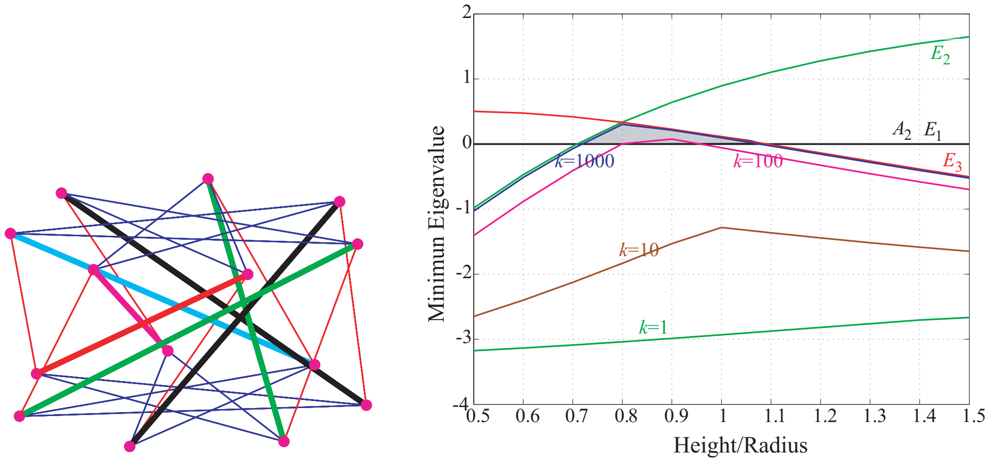


Fig. 6: The influence of the height/radius ratio and the stiffness/prestress ratio k on the stability of the structure $\mathbf{D}_7^{3,2}$.

Consider that the cables and struts have axial stiffness AE/l, and that the vertical cables carry a force density, due to the prestress, of q_v. We will consider the stiffness of an example structure for different values of $k = AE/lq_v$, where k is dimensionless. If the structure is linear-elastic, the strain due to a particular prestress will be $1/k$.

Figure 6 shows the smallest eigenvalues of the tangent stiffness matrix for the structure $\mathbf{D}_7^{3,2}$. Results are plotted for $k = 1, 10, 100, 1000$, and for the infinite stiffness case, where effectively $k \to \infty$. As k reduces, the structure becomes less stable, and eventually loses stability altogether.

5. CATALOGUE

Based on the methods described in this paper, we present in Table 2 a complete catalogue of prismatic tensegrity structures with dihedral symmetry for $n \leq 10$.

6. DISCUSSION AND CONCLUSION

A simple method for determining the self-equilibrated configuration of prismatic tensegrity structures with dihedral symmetry has been presented.

The conditions for the divisibility of prismatic tensegrity structures have been presented, based on the connectivity of horizontal and vertical cables. Divisible structures can be physically separated into several identical substructures.

Stability of prismatic tensegrity structures is demonstrated to be related to the connectivity of the horizontal and vertical cables, and is also sensitive to the height/radius ratio of the structure. It is also shown that stability of a tensegrity structure that is not super stable is also influenced by the selection of materials and the level of prestress.

A complete catalogue of the prismatic tensegrity structures with relative small number of members has been given.

The methods described in this paper have been implemented interactively, and can be accessed with the JAVA program online:
http://tensegrity.AIStructure.com/prismatic/

Acknowledgement:
The invaluable discussions with Prof. Bob Connelly of Cornell University are greatly appreciated.

Table 2: The stability of prismatic tensegrity structures $\mathbf{D}_n^{h,v}$. 's' denotes super stable, 'u' denotes unstable, and 'p' indicates that the structure is not super stable but is always prestress stable with arbitrary height/radius ratio. If the structure can be divided, its substructures are given; and if the structure is prestress stable only in a specific region of height/radius ratio from h_1 to h_2, then this region is presented by $[h_1, h_2]$.

$n = 3$		v: 1
h	1	s

$n = 4$		v: 1	2
h	1	s	u
	2	s	$2\mathbf{D}_2^{1,1}$

$n = 5$		v: 1	2
h	1	s	u
	2	s	u

$n = 6$		v: 1	2	3
	1	s	u	u
h	2	s	$2\mathbf{D}_3^{1,1}$	u
	3	s	p	$3\mathbf{D}_2^{1,1}$

$n = 7$		v: 1	2	3
	1	s	u	u
h	2	s	u	[0.75,1.05]
	3	s	u	u

$n = 6$		v: 1	2	3	4
	1	s	u	u	u
h	2	s	$2\mathbf{D}_4^{1,1}$	u	$2\mathbf{D}_4^{2,1}$
	3	s	[0.40,3.10]	u	u
	4	s	$2\mathbf{D}_4^{1,2}$	[0.35,2.35]	$4\mathbf{D}_2^{1,1}$

$n = 6$		v: 1	2	3	4
	1	s	u	u	u
h	2	s	u	u	u
	3	s	u	$3\mathbf{D}_3^{1,1}$	u
	4	s	u	[0.20,1.60]	u

$n = 6$		v: 1	2	3	4	
	1	s	u	u	u	u
	2	s	$2\mathbf{D}_5^{1,1}$	u	$2\mathbf{D}_5^{2,1}$	u
h	3	s	[0.70,1.35]	u	[0.75,1.25]	u
	4	s	$2\mathbf{D}_5^{1,2}$	u	$2\mathbf{D}_5^{2,2}$	u
	5	s	p	p	p	$5\mathbf{D}_2^{1,1}$

The first author is grateful for financial support from The Kyoto University Foundation and the EPSRC while visiting the Department of Engineering at the University of Cambridge.

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