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Topology optimization of trusses with stress and local constraints

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Abstract
A mixed integer programming formulation is presented for the truss topology optimization with stress and local constraints. Linear programming problems are successively solved based on a branch-and-bound method, where an upper-bound solution is obtained by solving nonlinear programming problems. It is shown in the examples that upper- and lower-bound solutions with small objective gap can be found, and the computational cost can be reduced by utilizing the local constraints.

1 Introduction

One of the main difficulties in topology optimization under stress constraints is that the constraints need not be satisfied by removed members (Sved and Ginos [4]). As a result, the optimal solution is often located at a cusp or a ridge of the feasible region. To overcome the difficulty due to discontinuities in the problem, several branch-and-bound-type methods have been presented (Ringertz [3]). A relaxation method has been presented by Cheng [1] for obtaining a good approximate solution. Another difficulty in topology optimization is that the solution often turns out to be an unrealistic design due to existence of unstable nodes, intersection of members, and existence of extremely slender members.

In this study, the topology optimization problem is first formulated as a Mixed Integer Programming (MIP) problem (Ohsaki and Katoh [2]). The local constraints on nodal instability and intersection of members are considered. The integer variables for indicating existence of nodes and members are used. A relaxed Linear Programming (LP) problem is solved to obtain a lower-bound solution. A NonLinear Programming (NLP) problem with fixed topology satisfying the local constraints is solved to find an upper-bound solution. It is shown in the examples that upper- and lower-bound solutions with small gap in objective value can be found by using the proposed branch-and-bound method.

2 Topology optimization problem.

2.1 Governing equations

Consider an elastic truss subjected to multiple static loads $\mathbf{P}^k$ $(k = 1, 2, \ldots, f)$. The equilibrium equation between $\mathbf{P}^k$ and the vector of axial forces $\mathbf{N}^k = \{N_i^k\}$ is given in the following form:

$$\mathbf{B}\mathbf{N}^k = \mathbf{P}^k, \quad (k = 1, 2, \ldots, f) \tag{1}$$

The stress σ_i^k of the ith member and N_i^k are obtained from the nodal displacements $\mathbf{U}^k$ as

$$\sigma_i^k = \frac{E}{L_i}\mathbf{B}_i^\top \mathbf{U}^k, \quad N_i^k = A_i \sigma_i^k, \quad (i = 1, 2, \ldots, m; k = 1, 2, \ldots, f) \tag{2}$$

where A_i and L_i are the cross-sectional area and the length of the ith member, respectively, E is the elastic modulus, $\mathbf{B}_i$ is the ith column of $\mathbf{B}$, m is the number of members.

2.2 Problem formulation

Let $y_i \in \{0, 1\}$ denote a variable indicating by $y_i = 1$ and $y_i = 0$, respectively, the existence and non-existence of the ith member in the initial ground structure. Stress constraints should be assigned only for members with $y_i = 1$. To avoid an unstable solution, the following constraints are assigned

$$A_i^{\mathrm{L}} y_i \le A_i \le A_i^{\mathrm{U}} y_i, \quad (i = 1, 2, \cdots, m) \tag{3}$$

where A_i^{U} and A_i^{L} are the upper bound and moderately large lower bound, respectively, for A_i. Note from (3) that $A_i = 0$ should be satisfied if $y_i = 0$.

Let $x_r \in \{0, 1\}$ be the variable indicating non-existence and possible existence of the rth node, respectively, by $x_r = 0$ and $x_r = 1$. The upper and lower bounds for the number of members connected to the rth node, if exists, are denoted by C_r^{U} and C_r^{L}, respectively. The set of indices of members connected to the rth node in the ground structure is denoted by J_r, and the constraints are given as

$$x_r C_r^{\mathrm{L}} \le \sum_{i \in J_r} y_i \le x_r C_r^{\mathrm{U}}, \quad (r = 1, 2, \ldots, s) \tag{4}$$

where s is the number of nodes including the supports. Note from (4) that $y_i = 0$ should be satisfied by all the members connected to a removed node with $x_r = 0$. $x_r = 1$ indicates existence of the rth node. The following constraints are to be satisfied for the pairs of mutually intersecting members S_i $(i = 1, \ldots, q)$.

$$\sum_{j \in S_i} y_j \le 1, \quad (i = 1, 2, \ldots, q) \tag{5}$$

Consider a problem of minimizing the total structural volume $V(\mathbf{A})$. The upper and lower bounds for σ_i^k are denoted by σ_i^{U} and σ_i^{L}, respectively. The topology optimization problem is then formulated as a mixed integer programming problem as

$$\begin{aligned}
\text{MIP}: \quad & \underset{\mathbf{A},\mathbf{y},\mathbf{x},\mathbf{U}^k,\boldsymbol{\sigma}^k,\mathbf{N}^k}{\text{minimize}} && V(\mathbf{A}) = \sum_{i=1}^{m} A_i L_i && (6)\\
& \text{subject to} && \sigma_i^{\mathrm{L}} y_i \le \sigma_i^k y_i \le \sigma_i^{\mathrm{U}} y_i, && (7)\\
& && y_i \in \{0, 1\}, \quad (i = 1, 2, \cdots, m) && (8)\\
& && x_r \in \{0, 1\}, \quad (r = 1, 2, \cdots, s) && (9)\\
& && \text{and } (1), (2), (3), (4), (5)
\end{aligned}$$

2.3 Lower- and upper-bound solutions

A relaxed problem of MIP is to be formulated as an LP to find the lower bound of the objective value V^{MIP} of MIP. The constraint (7) is rewritten by using variables N_i^k as

$$A_i \sigma_i^{\mathrm{L}} \le N_i^k \le A_i \sigma_i^{\mathrm{U}} \quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, f) \tag{10}$$

Note that (10) is satisfied for $0 \le y_i \le 1$ if (7) is satisfied. Hence, the relaxed LP of MIP is formulated as

$$\begin{aligned}
\text{LP}: \quad & \underset{\mathbf{A},\mathbf{y},\mathbf{x},\mathbf{N}^k}{\text{minimize}} && V(\mathbf{A})\\
& \text{subject to} && (1), (3), (4), (5), (10),\\
& && 0 \le y_i \le 1, \quad (i = 1, 2, \cdots, m) && (11)\\
& && 0 \le x_r \le 1, \quad (r = 1, 2, \cdots, s) && (12)
\end{aligned}$$

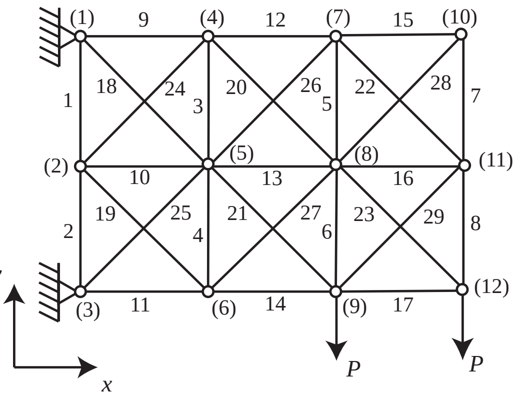

Figure 1: A 3×2 plane grid.

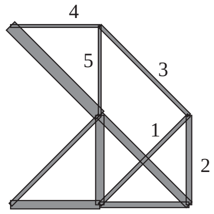

Figure 2: Initial LP solution for the 2×2 plane grid.

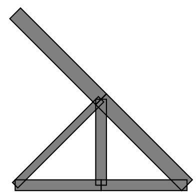

Figure 3: Initial upper-bound solution for the 2×2 plane grid.

Given a set $\mathcal{I}$ of existing members, the following NLP$_\mathcal{I}$ is defined.

$$\text{NLP}_\mathcal{I}: \quad \underset{\mathbf{A}}{\text{minimize}} \quad V(\mathbf{A}) = \sum_{i \in \mathcal{I}} A_i L_i \tag{13}$$

$$\text{subject to} \quad \sigma_i^{\mathrm{L}} \leq \sigma_i^k(\mathbf{A}) \leq \sigma_i^{\mathrm{U}}, \quad (i \in \mathcal{I}; k = 1, 2, \cdots, f) \tag{14}$$

$$A_i^{\mathrm{L}} \leq A_i \leq A_i^{\mathrm{U}}, \quad (i \in \mathcal{I}) \tag{15}$$

where $\sigma_i^k(\mathbf{A})$ is considered as function of $\mathbf{A}$. If NLP$_\mathcal{I}$ is feasible, its objective value $V^{\text{NLP}_\mathcal{I}}$ gives an upper bound of V^{MIP} since the solution of NLP$_\mathcal{I}$ satisfies all the constraints of MIP. The optimal solution of MIP is found by a branch-and-bound method, where LP and NLP$_\mathcal{I}$ are successively solved. Since NLP$_\mathcal{I}$ is not convex, is is not guaranteed that the globally optimal solution is always found.

3 Examples.

In the following examples, the units of force, length, area, volume and stress are kN, cm, cm^2, cm^3 and MPa, respectively. LP and NLP are solved by HOPDM Ver. 2.13 and NLPQL, respectively. Optimization has been carried out on Xeon 2.8GHz with 1GB memory.

Consider a plane truss grid as shown in Fig. 1. The lengths of the members in x- and y-directions are 200. Irrespective of the numbers of divisions, two loading conditions are considered, where the loads in the negative y-directions are applied at the node at the lowest end (node 12 in Fig. 1) and the node left to the lowest end (node 9 in Fig. 1), respectively. The magnitude of each load is 1000. The bounds for the stress are ± 200.0, and $C_r^{\mathrm{U}} = 6$. The value of C_r^{L} is 1 for the supports, 2 for the node at the lowest end, and 3 for the remaining nodes.

Optimal topology has been first found for the 2×2 grid. The LP solution at the first step is as shown in Fig. 2, where the width of a member is proportional to its cross-sectional area and $V^{\text{LP}} = 7.0000 \times 10^3$. To obtain an initial upper-bound solution, member 1 indicated in Fig. 2 is removed because it has smaller cross-sectional area in the pair of intersecting members. After removing member 1, the node connected by members 2 and 3 is removed because it violates the local constraint (4) with $C_r^{\mathrm{L}} = 3$. Hence, members 2 and 3 are removed, and members 4 and 5 are to be removed based on the local constraints. NLP$_\mathcal{I}$ is solved by fixing the topology to find an upper bound solution in Fig. 3, where $V^{\text{NLP}_\mathcal{I}} = 8.0000 \times 10^3$.

The branch-and-bound process is carried out to find the final upper-bound solution as shown in Fig. 4, where $V^{\mathrm{U}} = 7.9000 \times 10^3$. The optimization results are listed in the first row of Table 1, where *No. of steps* means the number of nodes of the branching tree. The final lower-bound solution is shown in Fig. 5, where $V^{\mathrm{L}} = 7.8000 \times 10^3$. Since this truss is statically indeterminate, the axial forces obtained by solving LP are not correct. Hence V^{L} has smaller value than V^{U} that was found by solving NLP$_\mathcal{I}$ only 5 times. However, the difference between V^{L} and V^{U} is very small. If we do not consider the local constraints, the numbers of steps, LP, and NLP are 529, 350, and 58, respectively, and CPU time is 5.05.

Table 1: Optimization results.

No. of division	m	n	A_i^{L}	A_i^{U}	No. of steps	No. of LP	No. of NLP	Upper bound V^{U}	Lower bound V^{L}	CPU time (s)
2×2	20	14	200	800	121	64	5	7.9000×10^3	7.8000×10^3	2.02
3×2	29	20	200	800	942	571	6	1.2900×10^4	1.2800×10^4	18.95
3×3	42	28	200	800	5874	3483	23	1.2467×10^4	1.2467×10^4	147.84
4×4	72	46	200	800	64890	42831	7	1.7067×10^4	1.7067×10^4	3072.84
4×4	72	46	200	600	68656	42707	73	1.8373×10^4	1.7916×10^4	2513.06
4×4	72	46	400	800	41001	26580	3	2.1507×10^4	2.1507×10^4	1794.08

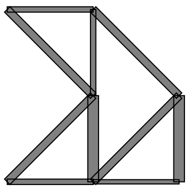

Figure 4: Final upper-bound solution for the 2×2 plane grid.

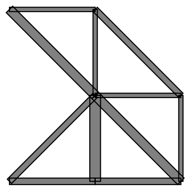

Figure 5: Final lower-bound solution for the 2×2 plane grid.

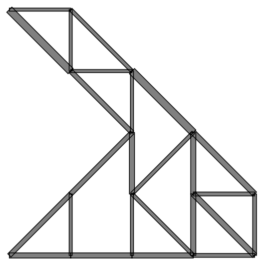

Figure 6: Final upper-bound solution for the 4×4 plane grid.

The optimization results for 3×2, 3×3 and 4×4 grids are also shown in Table 1, where the number of NLP steps is independent of the problem size, because it depends on the quality of the initial upper-bound solution. The final upper-bound solutions for 4×4 is shown Fig. 6. Note that $V^{\mathrm{L}} = V^{\mathrm{U}}$ is satisfied for 4×4, because the lower-bound solution is statically determinate. If A_i^{U} is decreased to 600, the optimization results are as shown in the fifth row of Table 1. The number of NLP steps is increased to 73. However, it is difficult to suggest a relation between the computational cost and the constraints or the size of the feasible region, because the CPU time for $A_i^{\mathrm{L}} = 400$ is almost half of that for $A_i^{\mathrm{L}} = 200$ as shown in the last row of Table 1.

4 Conclusions

A branch-and-bound method has been presented for obtaining upper- and lower-bound solutions of optimal topology of trusses under stress constraints. A rigorous problem formulation is first defined as a MIP problem with 0-1 variables indicating existence of nodes and members. The constraints on member intersection and nodal instability, which are called local constraints, are also considered. A moderately large lower bound is given for the cross-sectional area of an existing member. It has been shown that good upper and lower bounds can be found by using the proposed method. Computational cost can be drastically reduced by introducing local constraints to obtain practically acceptable optimal topologies.

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