# Form-Finding of Cable Domes with Specified Stresses by using Nonlinear Programming

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**Abstract**

A mathematical programming problem is proposed for form-finding of cable domes. The optimality conditions of the problem are derived to guarantee that the optimal solution coincides with the self-equilibrium configuration of the cable dome with specified member axial forces. The number of independent axial forces is investigated under the geometrical constraints as well as equilibrium conditions. An algorithm for designing cable domes is presented by using the primal-dual interior-point method. The self-equilibrium configurations are computed to demonstrate efficiency of the proposed algorithm.

## Introduction

Cable domes belong to a class of truss structures that cannot attain a stable equilibrium configuration without introducing prestresses to some members (Pellegrino, 1992). Tensegrities (Vilney, 1991; Murakami, 2001; Sultan *et al.*, 2001) and cable networks (Levy and Spillers, 1998) are included in the class of cable domes as special cases. In this paper, a nonlinear programming approach is proposed for initial form-finding of cable domes.

For form-finding problem of cable domes, Kawaguchi *et al.* (1999) proposed a least-square problem of nodal displacements with the specified external forces. Yuan and Dong (2002) presented a minimization problem of initial tention forces under stress constraints. As a pioneering work of form-finding, so-called force-density method was proposed by Schek (1974), which obtains the coordinates of internal nodes of the cable network with specified *force-density* of each member; i.e., the ratio of axial force to member length at the equilibrium state. However, from the practical point of view, it is strongly recommended to specify the axial forces directly, which has motivated the subsequent studies such as the smoothing method (Levy and Spillers, 1998).

We specify the member axial forces at the equilibrium state as well as the topology of the cable dome; i.e., the connectivity of cables and struts, and obtain the equilibrium configuration and initial length of each member. To this end, we propose a *nonlinear programming* (NLP) problem such that the optimal solution coincides with the equilibrium configuration with the specified axial forces.

For cable networks, which can be regarded as special cable domes without struts, the proposed optimization problem can be shown to be reduced to *second-order cone programming* (SOCP) problem (Ben-Tal and Nemirovski, 2001). SOCP is known as a special class of convex optimization problems, which is efficiently solvable by using the polynomial-time interior-point algorithms. The problem proposed for cable domes is nonconvex. However, it can be regarded as a natural extension of SOCP.

The set of member axial forces cannot be specified arbitrarily because (i) the axial forces should satisfy the equilibrium equations, and a set of axial forces is not necessarily realized by any configurations; (ii) the configurations of most cable domes actually built have some symmetry properties. In order to design the symmetric cable domes, we can conject that the same axial forces should be assigned to the symmetrically located members. These motivate us to investigate how to find an admissible set of axial forces. From equilibrium conditions with geometrical nonlinearity and the symmetry conditions based on the group representation theory (Ikeda and Murota, 2002), we derive a necessary and sufficient condition for the maximal subset of axial forces which can be specified arbitrarily.

## Form-finding problem of cable domes

Consider a cable dome in three dimensional space. Let $n^{\mathrm{m}}$ denote the number of members including both cables and struts. Assume that each cable and strut member can transmit only tension and compression forces, respectively. The subsets $\mathcal{I}_{\mathrm{C}}$ and $\mathcal{I}_{\mathrm{S}}$ of $\{1, \dots, n^{\mathrm{m}}\}$, respectively, are defined as the sets of all indices of cables and struts.

Let $n^{\mathrm{d}}$ denote the number of freedom of displacements. Consider the equilibrium state that was attained after introducing prestresses to some cables without external forces. Our objective is to obtain the coordinates of internal nodes $\boldsymbol{x} \in \Re^{n^{\mathrm{d}}}$ and the initial unstressed length $l_i^0$ of each member which satisfies the equilibrium conditions with the specified axial forces $q_i^*$ $(i = 1, \dots, n^{\mathrm{m}})$. As the first step, we specify the cross-sectional area $\bar{A}_i$ and the strain $\bar{\varepsilon}_i$ of the $i$th member, where

$$\bar{\varepsilon}_i = \begin{cases} \bar{\varepsilon} & (i \in \mathcal{I}_{\mathrm{C}}), \\ -\bar{\varepsilon} & (i \in \mathcal{I}_{\mathrm{S}}), \end{cases} \tag{1}$$

for a given $\bar{\varepsilon} > 0$, and formulate the form-finding problem as an NLP problem.

The standard Euclidean norm of vector $\boldsymbol{p} \in \Re^n$ is defined as $\|\boldsymbol{p}\| = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2}$. The member length $l_i$ at the equilibrium state can be written as

$$l_i = \|\boldsymbol{B}_i \boldsymbol{x} - \boldsymbol{d}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{2}$$

where $\boldsymbol{B}_i \in \Re^{3 \times n^{\mathrm{d}}}$ is a constant matrix determined only by the connectivity of nodes, and each of its elements is equal to either $\{-1, 0, 1\}$. $\boldsymbol{d}_i \in \Re^3$ is a constant vector that consists of the specified nodal coordinates of a support if the $i$th member is connected to the support, otherwise $\boldsymbol{d}_i = \boldsymbol{0}$.

For simplicity, we assume a linear elastic material, where the Young's modulus is denoted by $E$. Consider the following problem:

$$\begin{array}{lll} (\mathrm{D}(\bar{\varepsilon})): & \min & \displaystyle\sum_{i \in \mathcal{I}_{\mathrm{C}}} \frac{1}{2} E \bar{A}_i \bar{\varepsilon}^2 l_i^0 - \sum_{i \in \mathcal{I}_{\mathrm{S}}} \frac{1}{2} E \bar{A}_i \bar{\varepsilon}^2 l_i^0 \\ & \mathrm{s.t.} & (1 + \bar{\varepsilon}) l_i^0 \geq \|\boldsymbol{B}_i \boldsymbol{x} - \boldsymbol{d}_i\| \quad (i \in \mathcal{I}_{\mathrm{C}}), \\ & & (1 - \bar{\varepsilon}) l_i^0 \leq \|\boldsymbol{B}_i \boldsymbol{x} - \boldsymbol{d}_i\| \quad (i \in \mathcal{I}_{\mathrm{S}}), \end{array}$$

where $\boldsymbol{l}^0 = (l_i^0) \in \Re^{n^{\mathrm{m}}}$ and $\boldsymbol{x} \in \Re^{n^{\mathrm{d}}}$ are variables. Notice here that the objective function and the constraints of (D) correspond to the difference of the total strain energy of cables and struts, and the relaxed compatibility conditions, respectively.

**Lemma 1.** *Suppose* $\widetilde{l_i^0} > 0$ $(i = 1, \dots, n^{\mathrm{m}})$. $(\widetilde{\boldsymbol{l}^0}, \widetilde{\boldsymbol{x}}) \in \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{d}}}$ *is a local optimal solution of* (D) *only if there exists a* $\widetilde{\boldsymbol{q}} = (\widetilde{q}_i) \in \Re^{n^{\mathrm{m}}}$ *satisfying*

$$\widetilde{q}_i = E\bar{A}_i \bar{\varepsilon}_i \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{3}$$

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\top} \widetilde{q}_i \frac{\boldsymbol{B}_i \widetilde{\boldsymbol{x}} - \boldsymbol{d}_i}{\|\boldsymbol{B}_i \widetilde{\boldsymbol{x}} - \boldsymbol{d}_i\|} = \boldsymbol{0}, \tag{4}$$

$$(1 + \bar{\varepsilon}_i)\widetilde{l_i^0} = \|\boldsymbol{B}_i \widetilde{\boldsymbol{x}} - \boldsymbol{d}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{5}$$

*Proof.* Since (D) is a minimization problem and the objective function is an monotonic function of $l_i^0$, we see that all the constraints are active at any local optimal solution; i.e., the condition (5) is satisfied. From this and the Karush–Kuhn–Tucker conditions of (D), Lemma 1 is immediately obtained. $\square$

The following lemma guarantees that the (local) optimal solution of (D) satisfies the equilibrium conditions. This lemma plays a key role in the subsequent formulations.

**Lemma 2.** *Let* $(\widetilde{\boldsymbol{l}^0}, \widetilde{\boldsymbol{x}})$ *denote a local optimal solution of* (D). $\mathcal{C}(\widetilde{\boldsymbol{l}^0})$ *denotes the cable dome where the initial unstressed length of each member is specified as* $\widetilde{l_i^0}$ $(i = 1, \dots, n^{\mathrm{m}})$. *Then,* $\bar{\varepsilon}_i$ *and* $\widetilde{\boldsymbol{x}}$ *coincide with the member strain and the vector of coordinates of internal nodes of* $\mathcal{C}(\widetilde{\boldsymbol{l}^0})$*, respectively, at the equilibrium state.*

*Proof.* Let $\widetilde{\boldsymbol{l}^0}$ be constant. Consider the following problem:

$$\begin{aligned} (\mathrm{A}(\widetilde{\boldsymbol{l}^0})) : \quad & \min && \sum_{i \in \mathcal{I}_{\mathrm{C}} \cup \mathcal{I}_{\mathrm{S}}} \frac{1}{2} E\bar{A}_i \varepsilon_i^2 \widetilde{l_i^0} \\ & \text{s.t.} && (1 + \varepsilon_i)\widetilde{l_i^0} = \|\boldsymbol{B}_i \boldsymbol{x} - \boldsymbol{d}_i\| \quad (i \in \mathcal{I}_{\mathrm{C}} \cup \mathcal{I}_{\mathrm{S}}), \end{aligned}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_i) \in \Re^{n^{\mathrm{m}}}$ and $\boldsymbol{x} \in \Re^{n^{\mathrm{d}}}$ are variables; i.e., $(\mathrm{A}(\widetilde{\boldsymbol{l}^0}))$ is the minimization problem of the total potential energy for $\mathcal{C}(\widetilde{\boldsymbol{l}^0})$. Suppose $\widehat{\varepsilon}_i \neq -1$ $(i = 1, \dots, n^{\mathrm{m}})$. $(\widehat{\boldsymbol{\varepsilon}}, \widehat{\boldsymbol{x}}) \in \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{d}}}$ is a local optimal solution of $(\mathrm{A}(\widetilde{\boldsymbol{l}^0}))$ only if there exists a $\widehat{\boldsymbol{q}} = (\widehat{q}_i) \in \Re^{n^{\mathrm{m}}}$ satisfying

$$\widehat{q}_i = E\bar{A}_i \widehat{\varepsilon}_i \quad (i = 1, \dots, n^{\mathrm{m}}) \tag{6}$$

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i \widehat{q}_i \frac{\boldsymbol{B}_i \widehat{\boldsymbol{x}} - \boldsymbol{d}_i}{\|\boldsymbol{B}_i \widehat{\boldsymbol{x}} - \boldsymbol{d}_i\|} = \boldsymbol{0}, \tag{7}$$

$$(1 + \widehat{\varepsilon}_i)\widetilde{l_i^0} = \|\boldsymbol{B}_i \widehat{\boldsymbol{x}} - \boldsymbol{d}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{8}$$

From the stationary principle of the total potential energy, $\widehat{\varepsilon}_i$, $\widehat{q}_i$ and $\widehat{\boldsymbol{x}}$ satisfying (6)–(8) correspond to the strain and the axial force of each member, and the vector of internal nodes, respectively, at the equilibrium state of $\mathcal{C}(\widetilde{\boldsymbol{l}^0})$. By putting $\widetilde{\boldsymbol{q}} = \widehat{\boldsymbol{q}}$ and $\widetilde{\boldsymbol{x}} = \widehat{\boldsymbol{x}}$, we can see that Lemma 2 follows the fact such that (6)–(8) are equivalent to (3)–(5), which concludes the proof. $\square$

## On cable networks

Cable networks are included in the class of cable domes as the special case $\mathcal{I}_{\mathrm{S}} = \emptyset$. In this section, we investigate (D) and (A) for cable networks. Let $\boldsymbol{v}_i \in \Re^3$ and $\boldsymbol{v} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n^{\mathrm{m}}}) \in \Re^{3n^{\mathrm{m}}}$.

**Lemma 3.** *Suppose* $\mathcal{I}_{\mathrm{S}} = \emptyset$. $(\widetilde{\boldsymbol{l}^0}, \widetilde{\boldsymbol{x}}) \in \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{d}}}$ *is a global optimal solution of* (D) *if and only if there exists a* $(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{v}}) \in \Re^{n^{\mathrm{m}}} \times \Re^{3n^{\mathrm{m}}}$ *satisfying*

$$\widetilde{q}_i = E\bar{A}_i\bar{\varepsilon}_i, \quad \widetilde{q}_i \geq \|\widetilde{\boldsymbol{v}}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{9}$$

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\top} \widetilde{\boldsymbol{v}}_i = \boldsymbol{0}, \tag{10}$$

$$(1 + \bar{\varepsilon})\widetilde{l_i^0} \geq \|\boldsymbol{B}_i\widetilde{\boldsymbol{x}} - \boldsymbol{d}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{11}$$

*Proof.* This lemma follows the KKT conditions for the nondifferentiable convex optimization problem (Rockafellar, 1970, Theorem 31.3) and the self-duality of the second-order cone (Ben-Tal and Nemirovski, 2001). $\square$

Notice here that Lemma 3 states the necessary and sufficient conditions for global optimality, whereas Lemma 1 states the necessary conditions for local optimality. Moreover, we see that (D) for $\mathcal{I}_{\mathrm{S}} = \emptyset$ is an SOCP problem (Ben-Tal and Nemirovski, 2001), which is a special class of convex optimization problems. The authors showed that the equilibrium shapes and member initial lengths of cable networks can be obtained by solving that SOCP problem by using the primal-dual interior-point method (Kanno and Ohsaki, 2002).

We also showed that, for $\mathcal{I}_{\mathrm{S}} = \emptyset$, (A) can be reformulated into the following convex problem:

$$(\mathrm{A_C}(\widetilde{\boldsymbol{l}^0})): \quad \min \quad \sum_{i \in \mathcal{I}_{\mathrm{C}}} \frac{1}{2} E\bar{A}_i\varepsilon_i^2 \widetilde{l_i^0}$$
$$\text{s.t.} \quad (1 + \varepsilon_i)\widetilde{l_i^0} \geq \|\boldsymbol{B}_i\boldsymbol{x} - \boldsymbol{d}_i\| \quad (i \in \mathcal{I}_{\mathrm{C}}),$$

which is also embedded into an SOCP problem. See, Kanno *et al.* (2002) for more details.

## Maximal set of independent axial forces

In (D), we cannot specify all $\bar{A}_i$s arbitrary. Let $\beta$ denote the maximal number of independent $\bar{A}_i$, or $q_i^*$. Note that the most of cable domes actually built have some symmetry properties, e.g., as shown in Fig.1. Hence, we have $\beta < n^{\mathrm{m}}$ because (i) the axial forces should satisfy the equilibrium equations, and a set of axial forces is not necessarily realized by arbitrary configurations; (ii) in accordance with the symmetry of configuration, the distribution of axial forces should also have some symmetry property.

Letting $\boldsymbol{v}_i \in \Re^3$ denote the internal force vector of the $i$th member, we have $q_i = EA_i\bar{\varepsilon} = \|\boldsymbol{v}_i\|$. The equilibrium equations are written as

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\top} \boldsymbol{v}_i = \boldsymbol{0}, \tag{12}$$

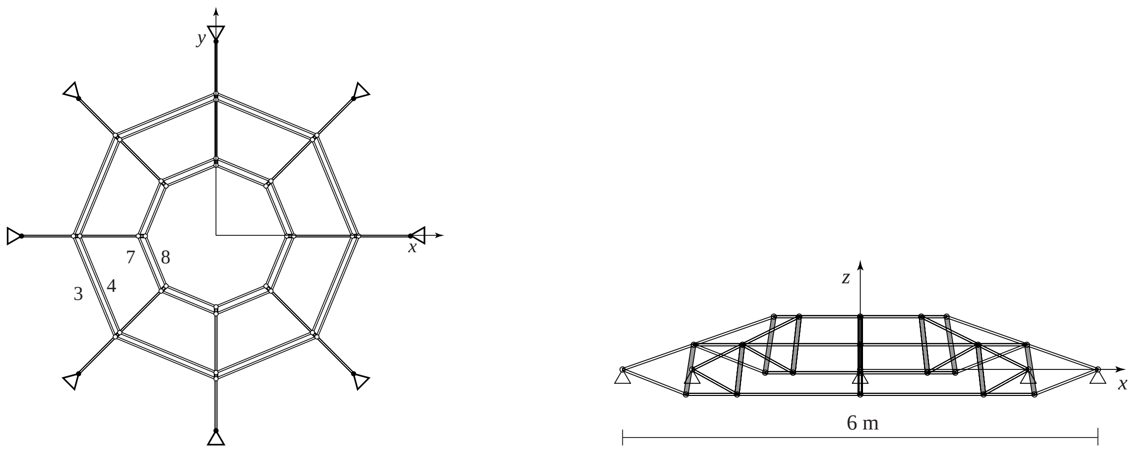


Figure 1: Cable dome.

which are valid under finite deformation. We write $\boldsymbol{B} = [\boldsymbol{B}_1^\top, \dots, \boldsymbol{B}_{n^{\mathrm{m}}}^\top]^\top \in \Re^{3n^{\mathrm{m}} \times n^{\mathrm{d}}}$ for simplicity. Then (12) is written as $\boldsymbol{B}^\top \boldsymbol{v} = \boldsymbol{0}$. Under assumptions such that the cable dome is simply connected, we can show $\mathrm{rank}\, \boldsymbol{B} = n^{\mathrm{d}}$ (Kanno and Ohsaki, 2003). Hence, we have $\beta = 3n^{\mathrm{m}} - n^{\mathrm{d}}$ if the constraints on symmetry of configuration is not included.

Suppose that we make the constraint such that the configuration of the cable dome should have some symmetry. For each $i = 1, \dots, n^{\mathrm{m}}$, define $\widetilde{\boldsymbol{h}}_i \in \Re^3$ and $\widetilde{l}_i \in \Re$ by

$$\widetilde{\boldsymbol{h}}_i = \boldsymbol{B}_i \widetilde{\boldsymbol{x}} - \boldsymbol{d}_i, \quad \widetilde{l}_i = \|\widetilde{\boldsymbol{h}}_i\|, \tag{13}$$

where $\widetilde{\boldsymbol{h}}_i$ and $\widetilde{l}_i = (1 + \bar{\varepsilon})\widetilde{l}_i^0$ correspond to the deformation vector and the length of the $i$th member at the optimal solution of (D). The invariance condition of configuration; i.e., the symmetry property of configuration, can be written as

$$\boldsymbol{C}^\top \widetilde{\boldsymbol{h}} = \boldsymbol{0}, \tag{14}$$

where $\boldsymbol{C} \in \Re^{3n^{\mathrm{m}} \times 3n^{\mathrm{g}}}$ is an appropriate constant matrix and $\widetilde{\boldsymbol{h}} = (\widetilde{\boldsymbol{h}}_1, \dots, \widetilde{\boldsymbol{h}}_{n^{\mathrm{m}}}) \in \Re^{3n^{\mathrm{m}}}$. Letting $G$ denote the finite group which labels the symmetry of $\boldsymbol{h}$, $n^{\mathrm{g}}$ is the positive integer determined from the number of all inequivalent irreducible unitary representations of $G$ (See, e.g, Ikeda and Murota (2002) for the basic buckground of the group representation theory). It follows from the second equation in (13) that (14) can be rewritten as

$$\boldsymbol{C}_A^\top \widetilde{\boldsymbol{l}} = \boldsymbol{0}, \tag{15}$$

where $\boldsymbol{C}_A \in \Re^{n^{\mathrm{m}} \times n^{\mathrm{g}}}$ is constant, and $\widetilde{\boldsymbol{l}} = (\widetilde{l}_i) \in \Re^{n^{\mathrm{m}}}$. The distribution of member cross-sectional areas should also have the symmetry property, which is written by using the same matrix $\boldsymbol{C}_A$ in (15) as

$$\boldsymbol{C}_A^\top \bar{\boldsymbol{A}} = \boldsymbol{0}, \tag{16}$$

where $\bar{\boldsymbol{A}} = (\bar{A}_i) \in \Re^{n^{\mathrm{m}}}$. Recall that we specify the strains $\bar{\varepsilon}_i$ as (1), and $\boldsymbol{v}_i = E\bar{A}_i\bar{\varepsilon}_i\boldsymbol{h}_i/\|\boldsymbol{h}_i\|$. By using the symmetry of $\bar{\boldsymbol{A}}$, (14) is reduced to

$$\boldsymbol{C}^\top \widetilde{\boldsymbol{v}} = \boldsymbol{0}. \tag{17}$$

Accordingly, $\boldsymbol{v}$ should satisfy the system of (12) and (17), from which it follows that

$$\beta = 3n^{\mathrm{m}} - \mathrm{rank}(\boldsymbol{B}, \boldsymbol{C})^{\top} \tag{18}$$

There exists a set of independent $\mathrm{rank}(\boldsymbol{B}, \boldsymbol{C})^{\top}$ column vectors in the matrix $(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}})^{\top}$, and we can choose $\beta$ column vectors that do not belong to that set. Hence, we have $\beta$ members corresponding to the selected column vectors, and $\bar{A}_i$ or $q_i^*$ can be specified arbitrary for these members.

Suppose that we specify the axial force $q_i^*$ and the cross-sectional area $A_i^*$ of each member. Then, the strain $\varepsilon_i^*$ at the equilibrium state should satisfy

$$q_i^* = EA_i^* \varepsilon_i^* \quad (i = 1, \dots, n^{\mathrm{m}}),$$

which implies that specifying $A_i^*$ and $q_i^*$ is equivalent to specifying $A_i^*$ and $\varepsilon_i^*$. Moreover, we can always choose $\bar{A}_i > 0$ and $\bar{\varepsilon} > 0$ such that

$$A_i^* |\varepsilon_i^*| = \bar{A}_i \bar{\varepsilon} \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{19}$$

Accordingly, the equilibrium configuration $\boldsymbol{x}$ of the cable dome with the specified axial forces $\boldsymbol{q}^*$ can be obtained by using the following algorithm:

**Algorithm 4.**

**Step 1:** Set the topology $\boldsymbol{B}_i$, $\mathcal{I}_{\mathrm{C}}$ and $\mathcal{I}_{\mathrm{S}}$, the coordinates of supports $\boldsymbol{d}_i$, and the symmetry property $\boldsymbol{C}_A$ and $\boldsymbol{C}$.

**Step 2:** Compute $\beta$ from (18).

**Step 3:** Specify $q_i^*$; i.e., the pairs of $A_i^*$ and $\varepsilon_i^*$ of $\beta$ members.

**Step 4:** For the given $\bar{\varepsilon} > 0$, compute $\bar{A}_i$ of $\beta$ members from (19).

**Step 5:** Compute $\bar{A}_i$ of the remaining $n^{\mathrm{m}} - \beta$ members from (16) and the equilibrium equations.

**Step 6:** Compute a solution of (D) by using the interior-point method.

In Step 5 of Algorithm 4, we compute the unknown $\bar{A}_i$s from (16) after substituting the $\bar{A}_i$s of $\beta$ members obtained in Step 4. Although we prepare the matrix $(\boldsymbol{B}, \boldsymbol{C})^{\top}$ in Step 2, the unknown $\bar{A}_i$s can be obtained by solving the small scale equilibrium equations at a node connected by each member with unknown $\bar{A}_i$; i.e. it is not required to solve the large scale linear equations $(\boldsymbol{B}, \boldsymbol{C})^{\top}\boldsymbol{v} = \boldsymbol{0}$ in Step 5.

## Examples

Consider the cable dome as shown in Fig.1 that has 16 struts. We specify the coordinates of supports, whereas the coordinates of internal nodes are unknown. Let $s$ and $r(\varphi)$, respectively, denote the reflection with respect to the $xz$-plane and the counter-clockwise rotation around the $z$-axis with the angle $\varphi$. We make the constraints such that the geometry of this cable dome should be symmetric

Table 1: Specified values of $\bar{A}_i^*$.

| member | $\bar{A}_i^*$ (cm$^2$) | member | $\bar{A}_i^*$ (cm$^2$) | member | $\bar{A}_i^*$ (cm$^2$) |
|---|---|---|---|---|---|
| 2 | 30.0 | 6 | 3.0 | 13 | 10.0 |
| 3 | 35.0 | 7 | 3.5 | 14 | 1.0 |
| 4 | 28.0 | 8 | 3.5 | | |

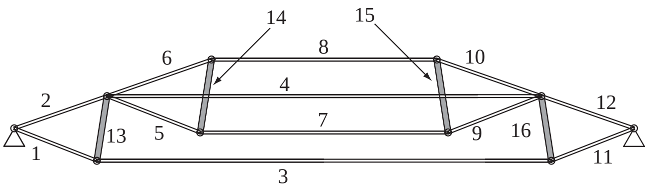


Figure 2: Plane element.

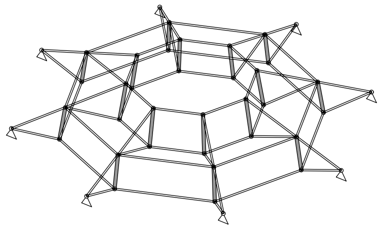

Figure 3: Equilibrium configuration.

with respect to any transformation by the element of the dihedral group of degree 8 defined as

$$\mathrm{D}_8 = \big\{ r(\pi k/4),\ sr(\pi k/4) \big| (k = 1, \dots, 8) \big\}.$$

From the rotational symmetry of the dome, we only have to consider the plane element as shown in Fig.2, where $n^{\mathrm{m}} = 16$, $n^{\mathrm{d}} = 16$, $\mathcal{I}_{\mathrm{C}} = \{1, \dots, 12\}$ and $\mathcal{I}_{\mathrm{S}} = \{13, \dots, 16\}$. The tension forces of the members 3, 4, 7, and 8 in Fig.1, which are denoted by $q_i^h$ $(i = 3, 4, 7, 8)$, are modeled by those of members 3, 4, 7 and 8 in Fig.2 as

$$q_i = 2q_i^h \sin\left(\frac{\pi}{8}\right) \quad (i = 3, 4, 7, 8).$$

Then the symmetry constraints are reduced to those with respect to $z$-axis on the model shown in Fig.2, from which we obtain $\mathrm{rank}(\boldsymbol{B}, \boldsymbol{C})^\top = 24$. By applying (18) to the two-dimensional model, we see $\beta = 2n^{\mathrm{m}} - \mathrm{rank}(\boldsymbol{B}, \boldsymbol{C})^\top = 8$. It is easily verified that we can choose the set of 8 members as $\{2, 3, 4, 6, 7, 8, 13, 14\}$. Suppose the cross-sectional areas are assigned as listed in Tab.1. From the equilibrium equations, $\bar{A}_1 = 27.1141$ and $\bar{A}_5 = 2.7114$ are obtained, whereas $\bar{A}_i$ $(i \in \{9, \dots, 12\})$ are immediately obtained from the symmetry conditions.

Letting $E = 205.8$ GPa and $\bar{\varepsilon} = 1.0 \times 10^{-3}$, the problem (D) has been solved by using NUOPT (1998), which is an implementation of primal-dual interior-point method for NLP. The obtained equilibrium configuration is as shown in Fig.3. At each node, the norm of unbalanced force is within $10^{-6}$ of the average norm of axial forces, which illustrates the accuracy of the proposed method.

## Conclusions

We have formulated the form-finding problem of cable domes as an NLP problem with the specified axial forces, which can be regarded as the minimization problem of the difference of the total strain energy between cable members and struts under constraints on compatibility conditions. By investigating the optimality conditions, it has been guaranteed that the optimal solution of the proposed NLP problem satisfies the equilibrium conditions. We have also proposed the method to find a maximal set of independent axial forces explicitly considering the specified symmetry property of the geometry of the structure.

It has been demonstrated in the numerical example that the set of admissible axial forces can be found by using the proposed algorithm. Since we can solve the proposed NLP promlem by the well-developed existing softwares of NLP based on primal-dual interior-point method, our task is only to input geometrical and material information of cable domes, and no effort is required to develop any analysis software.

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