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Critical Minor Imperfection Corresponding to Stable Bifurcation

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Abstract

A simple and computationally inexpensive approach is presented for obtaining the most sensitive imperfection mode corresponding to the maximum load factor of the stable bifurcation point. The critical point of an imperfect system is found by solving an anti-optimization problem, where the load factor is minimized with respect to the imperfection parameters and the nodal displacements under constraint on the lowest eigenvalue of the stability matrix. It is shown in the examples that a minor imperfection that is usually dismissed is very important in evaluating the maximum load of a flexible structure.

Keywords: Buckling analysis; Imperfection sensitivity; Minor imperfection; Anti-optimization; Convex model, Simultaneous analysis and optimization

Introduction

The lower bound of the maximum load factor of a geometrically nonlinear structure that exhibits bifurcation-type instability may be evaluated based on the most critical mode of imperfection that maximizes the reduction of the load carrying capacity under constraint on the norm of the imperfection. There have been several studies for finding the most critical mode of imperfection for simple and coincident unstable symmetric bifurcation points (Ho, 1974, Ikeda and Murota, 1990) based on a perturbation approach. Ohsaki *et al.* (1998) presented an optimization method considering the reduction of the maximum load factor due to the most critical mode of major imperfection; e.g. antisymmetric imperfection of a symmetric system.

Contrary to imperfection-sensitive structures such as cylindrical shells and stiffened plates, the bifurcation point of a column that has a stable postbuckling path disappears due to a small major imperfection. In this paper, a simple and numerically inexpensive approach is presented for determining the maximum load factors of imperfect elastic structures considering imperfections of nodal locations and nodal loads. An anti-optimization problem is formulated so as to minimize the bifurcation load factor within the convex bounds on the imperfection parameters. A relaxed problem of simultaneous analysis and design is solved to determine the bifurcation load by minimizing the load factor under constraint on the lowest eigenvalue of the stability matrix allowing imperfections of nodal loads. This way, laborious nonlinear path-following analysis is avoided. It is shown in the examples of a 20-bar truss that the most critical mode of minor imperfection can be successfully obtained by the proposed approach.

Maximum load factor of an imperfect system

Consider a finite dimensional elastic structure subjected to quasi-static proportional loads $\mathbf{P} = \Lambda \mathbf{p}$, where Λ is the load factor. The vector of nodal displacements is denoted by $\mathbf{U} = \{U_i\}$. The total potential energy is defined as $\Pi(\mathbf{U}, \Lambda; \xi) = H(\mathbf{U}; \xi) - \Lambda \mathbf{p}^T(\xi)\mathbf{U}$, where $H(\mathbf{U}; \xi)$ is the strain energy and ξ is the imperfection parameter. The stability matrix $\mathbf{S}$, which is the tangent stiffness matrix is defined as the Hessian of H with respect to $\mathbf{U}$. Let λ_r and $\Phi_r = \{\Phi_{ri}\}$ denote the rth eigenvalue and eigenvector of $\mathbf{S}$. The critical load factor Λ^{c} corresponds to $\lambda_1 = 0$, where λ_1 is the lowest eigenvalue. Define α as

$$\alpha = \sum_{i=1}^{n} \frac{\partial^2 \Pi}{\partial \xi \partial U_i} \Phi_{1i} \tag{1}$$

where Φ_{1i} is the ith component of Φ_1, and n is the number of degrees of freedom. The major and minor imperfections are characterized by $\alpha \neq 0$ and $\alpha = 0$, respectively (Roorda, 1968). For a symmetric system, a symmetric and antisymmetric imperfections correspond to minor and major imperfections, respectively (Ohsaki, 2000, Ohsaki, 2001).

Since antisymmetric components of deformation along the bifurcation path of the stable bifurcation point may be very large, the maximum load should be defined in view of the stresses and/or displacements. The most critical imperfection is defined by reduction of the bifurcation load factor due to minor imperfections based on the following reasons:

1. Even for a stable bifurcation, reaching the bifurcation point should be avoided because it leads to a sudden dynamic deformation. Since the bifurcation point disappears if a major imperfection exists, the most critical imperfection for this case is a minor imperfection.

2. Since we consider a flexible structure and allow moderately large deformation, the maximum load defined by deformation constraints is dramatically reduced by minor imperfections rather than major imperfections, and sensitivity of the maximum load is almost equivalent to that of the bifurcation point.

Anti-optimization problem

Let $\boldsymbol{\xi}_i$ $(i = 1, 2, \ldots, m)$ denote the vector of ith set of imperfection parameters. The norm of $\boldsymbol{\xi}_i$ is denoted by $e_i(\boldsymbol{\xi}_i)$ which is a convex function of $\boldsymbol{\xi}_i$. An upper bound

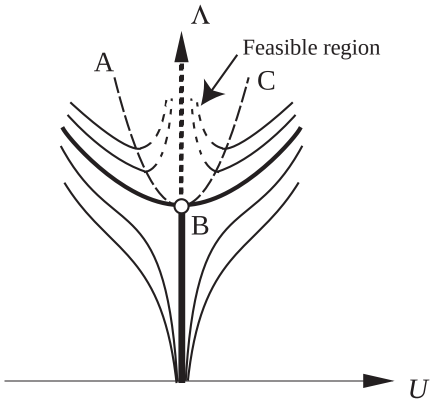

Figure 1. Feasible region for the eigenvalue constraint.

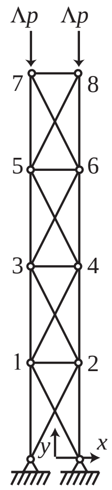

Figure 2. A column-type 20-bar plane truss.

$\bar{e}_i$ is given for $e_i(\boldsymbol{\xi}_i)$ by an approach similar to that of the convex model (Ben-Haim and Elishakoff, 1990). The set of vectors $\boldsymbol{\xi}_i$ is divided into major imperfections $\boldsymbol{\xi}_i^{\mathrm{I}}$ and minor imperfections $\boldsymbol{\xi}_i^{\mathrm{II}}$. The values corresponding to major and minor imperfections are indicated by superscripts $(\ )^{\mathrm{I}}$ and $(\ )^{\mathrm{II}}$, respectively; e.g. the upper bound for $e_i^{\mathrm{II}}(\boldsymbol{\xi}_i^{\mathrm{II}})$ is denoted by $\bar{e}_i^{\mathrm{II}}$. Let $\boldsymbol{\xi}^{\mathrm{II}}$ denote the vector that consists of all the elements of the vectors $\boldsymbol{\xi}_i^{\mathrm{II}}$ $(i = 1, 2, \dots, m^{\mathrm{II}})$.

If we fix $\boldsymbol{\xi}^{\mathrm{II}}$ and only consider major imperfections, the region in the $(\Lambda - U)$-space where $\lambda_1 \leq 0$ is satisfied is as indicated by *feasible region* in Fig. 1, where U is a representative generalized displacement generated due to existence of a major imperfection. The thick curve in Fig. 1 is the bifurcation path of the perfect system, and thin curves are equilibrium paths of imperfect systems. The dotted curves indicate unstable equilibrium points. Note that the region bounded by the dashed curve ABC is feasible for the constraint $\lambda_1 \leq 0$. Since we consider the case where the perfect system exhibits stable bifurcation, the feasible region in the vicinity of the bifurcation point is convex with respect to U and Λ. Hence, the buckling load factor is found by minimizing Λ with respect to $\boldsymbol{\xi}_i^{\mathrm{I}}$ under constraint of $\lambda_1 \leq 0$. We further minimize Λ with respect to $\boldsymbol{\xi}_i^{\mathrm{II}}$ to obtain the most sensitive imperfection. Since both processes correspond to minimization of Λ, these processes can be carried out simultaneously.

Since $\mathbf{U}$ is considered as variables that are the same level as $\boldsymbol{\xi}^{\mathrm{II}}$, the exact equilibrium state corresponding to the *perfect* nodal loads is not needed. Let $\boldsymbol{\xi}$ denote the vector consisting of $\boldsymbol{\xi}_i$ including minor and major imperfections. The internal nodal forces $\mathbf{F}^*(\mathbf{U};\boldsymbol{\xi}) = \{F_j^*(\mathbf{U};\boldsymbol{\xi})\}$ equivalent to the displacements $\mathbf{U}$ of an imperfect system are defined by

$$F_j^*(\mathbf{U};\boldsymbol{\xi}) = \frac{\partial H(\mathbf{U};\boldsymbol{\xi})}{\partial U_j}, \quad (j = 1, 2, \dots, n) \tag{2}$$

where $(\ )^*$ indicates a function of $\mathbf{U}$ and $\boldsymbol{\xi}$. F_j^* is then calculated for each trial displacement vector during the optimization process.

The maximum load factor is obtained by solving the following anti-optimization

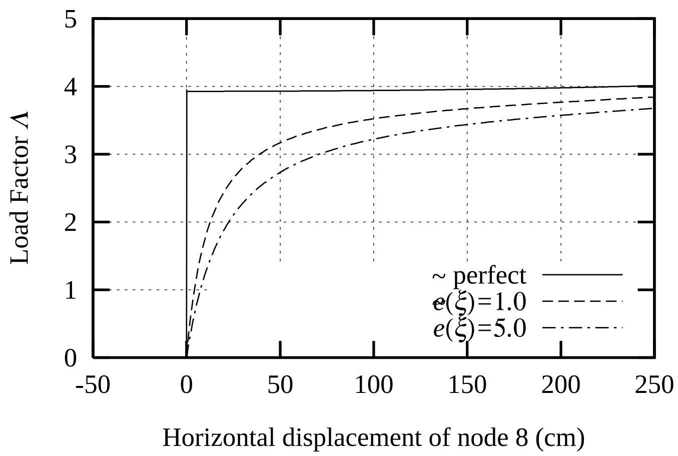

Figure 3. Relation between δ and Λ for perfect and imperfect systems in the direction of Φ^A.

problem (Elishakoff et al., 1994) for finding the minimum value of Λ under constraints on the norms of imperfections and the lowest eigenvalue of the stability matrix:

$$\begin{aligned} \text{AOP: minimize} \quad & \Lambda \\ \text{subject to} \quad & e_i(\boldsymbol{\xi}_i) \leq \bar{e}_i, \ \ (i = 1, 2, \ldots, m) && (3) \\ & \Lambda(p_j - \Delta p) \leq F_j^*(\mathbf{U}; \boldsymbol{\xi}) \leq \Lambda(p_j + \Delta p), \quad (j = 1, 2, \ldots, n) && (4) \\ & \lambda_1^*(\mathbf{U}; \boldsymbol{\xi}) \leq 0 && (5) \end{aligned}$$

The variables of AOP are $\mathbf{U}, \boldsymbol{\xi}$ and Λ. Only computation of $F_i^*(\mathbf{U}; \boldsymbol{\xi})$ and $\lambda_1^*(\mathbf{U}; \boldsymbol{\xi})$ is needed for the current value of $\mathbf{U}$ and $\boldsymbol{\xi}$ at each iterative step of optimization, and the laborious path-following analysis is not needed.

Examples

Consider a column-type 20-bar plane truss as shown in Fig. 2. The lengths of members in x- and y-directions are 100 cm and 200 cm, respectively. The cross-sectional areas are 2.0 cm^2 for all the members, and $p = 98$ kN. The elastic modulus is 205.8 GPa. The sequential quadratic programming is used for optimization, and the gradients are computed by the finite difference approach. Computation has been carried out on a personal computer with AMD Athron 1.0 GHz. The vector $\boldsymbol{\xi}$ of the imperfection parameters consists of the coordinates of all the nodes except two supports. The norm $\tilde{e}(\boldsymbol{\xi})$ of the imperfection is defined as $\tilde{e}(\boldsymbol{\xi}) = \frac{1}{n}\sqrt{\boldsymbol{\xi}^T\boldsymbol{\xi}}$, and Δp is 1% of p. Let Φ^{A} and Φ^{S} denote the lowest antisymmetric and symmetric linear buckling modes, respectively, of the perfect system.

The critical load factor of the perfect system is 3.9366, where the buckling mode is antisymmetric with respect to the y-axis and the critical point is a symmetric bifurcation point. Imperfection sensitivity properties are first investigated for imperfections in the directions of Φ^{A} and Φ^{S} which correspond to major and minor imperfections, respectively. Note that the imperfection mode $\Delta\mathbf{p}$ of the

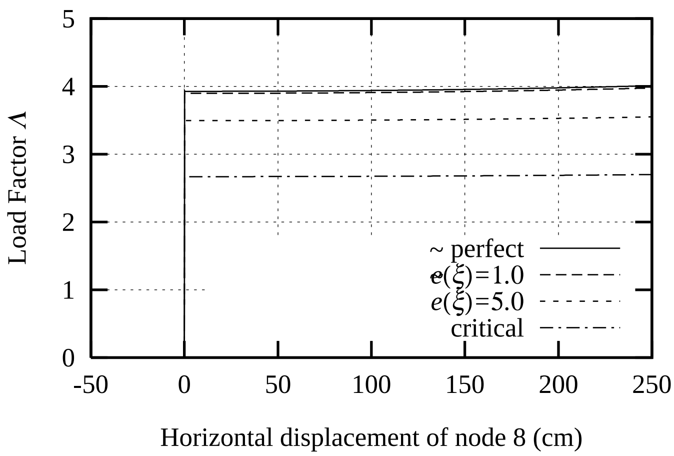

Figure 4. Relation between δ and Λ for perfect and imperfect systems in the direction of Φ^S.

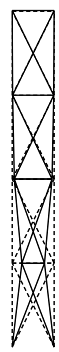

Figure 5. Most critical mode of imperfection.

Table 1. Number of iteration steps, CPU time and the objective value.

	Number of steps	CPU time (sec.)	Λ
Symmetric initial solution	38	36.8	2.6693
Symmetric imperfection	14	6.7	2.6693
Asymmetric initial solution	27	25.2	2.6693
Linear strain	29	26.5	2.6864
Incremental analysis	67	54.2	2.6708

nodal loads is also considered in the same directions as the nodal imperfections, where $\Delta\mathbf{p}$ is scaled so that its maximum absolute value is equal to 1% of p. Fig. 3 shows the relation between the horizontal displacement δ of node 8 and the load factor for three cases of perfect and imperfect systems in the direction of Φ^A with $\tilde{e}(\boldsymbol{\xi}) = 1.0$ and 5.0 cm. Fig. 4 shows the relation between δ and Λ for minor imperfection corresponding to Φ^{S}.

Suppose the maximum load factor Λ^{M} is defined by the displacement constraint $\delta \leq \bar{\delta}$. It may be observed from Figs. 3 and 4 that the reduction of Λ^{M} due to a major imperfection is larger than that to a minor imperfection if $\bar{\delta}$ is small, but a minor imperfection dominates if $\bar{\delta}$ is sufficiently large. For instance, if $\tilde{e}(\boldsymbol{\xi}) = 5.0$ cm, reduction in the direction of Φ^{S} is larger than that of Φ^{A} in the range $\delta > 179$ cm. The important property observed in Fig. 4 is that the magnitude of reduction of Λ^{M} does not strongly depend on the value of $\bar{\delta}$. Therefore, the most critical mode of minor imperfection may be successfully obtained by solving AOP considering only the bifurcation load factor.

The most critical mode of nodal imperfection $\boldsymbol{\xi}^{\mathrm{M}}$ obtained from a symmetric initial solution is symmetric as shown in Fig. 5. The number of optimization steps, CPU time and the optimal objective value are as listed in the first row of Table 1. The relation between δ and Λ for the most critical case is also plotted in Fig. 4. If we consider only symmetric components of $\boldsymbol{\xi}$ and $\mathbf{U}$, the maximum load factor of the symmetric system is 2.6693 which agrees within the accuracy of five digits with the value obtained by the formulation including asymmetric imperfections and deformations. The mean absolute value of deviation of $\boldsymbol{\xi}^{\mathrm{M}}$ from those of the symmetric system is 4.3335×10^{-3} which is very small compared to the maximum absolute value 13.7810 of $\boldsymbol{\xi}^{\mathrm{M}}$. The computational cost is reduced as shown in the second row of Table 1 if we consider only symmetric imperfections and deformations.

If we do not exclude major imperfections, the deviation from the symmetric solution increases to 3.4390×10^{-2} which is still very small. The computational cost is smaller, as shown in the third column of Table 1, than that from the symmetric initial solution. The maximum load factor obtained by linearizing the equilibrium equation (2) with respect to $\mathbf{U}$ is 2.6864. The convergence property, however, does not improve as the result of neglecting the geometrical nonlinearity as observed from the fourth row of Table 1. If the critical point is found by tracing the fundamental equilibrium path, the computational results are as listed in the last row of Table 1 which should be compared to the second row because only symmetric imperfections are considered here. Note that the computational cost for this case is very large compared to that for AOP.

Conclusions

A simple and computationally inexpensive approach has been presented for obtaining the most sensitive imperfection mode corresponding to the maximum load factor of the stable bifurcation point. It has been shown that a minor imperfection can be more critical than a major imperfection if a moderately large deformation is allowed. The most critical minor imperfection has been successfully obtained by solving the proposed anti-optimization problem. The anti-optimal solutions have been found under several problem settings, and it has been confirmed that the proposed method has advantages over the method with path-following analysis in view of computational cost and convergence property.

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