Symmetry of the Solution of Semidefinite Program by Using Primal-Dual Interior-Point Method

Yoshihiro Kanno, Makoto Ohsaki and Naoki Katoh

Department of Architecture and Architectural Systems, Kyoto University, Kyoto 606-8501, Japan
kanno@is-mj.archi.kyoto-u.ac.jp, {ohsaki, naoki}@archi.kyoto-u.ac.jp

1. Abstract

Symmetry of an optimal solution of Semi-Definite Program (SDP) is discussed based on symmetry property of the central path that is traced by a primal-dual interior-point method. A *symmetric SDP* is defined by operators for rearranging elements of matrices and vectors, and the solution on the central path is proved to be symmetric. Therefore, it is theoretically guaranteed that a symmetric optimal solution is always obtained by using a primal-dual interior-point method. The optimization problem of symmetric trusses under eigenvalue constraints is shown to be formulated as a symmetric SDP. Numerical experiments illustrate convergence to strictly symmetric optimal solutions.

2. Keywords: semidefinite program, primal-dual interior-point method, structural optimization, eigenvalue optimization

3. Introduction

Symmetry of an optimal solution has great significance from practical point of view especially in structural optimization. Since most of structures actually constructed have some symmetry properties, it is desired to obtain a symmetric design as a result of optimization. Consider the truss optimization problem to find optimal cross-sectional areas for a given configuration. To obtain the symmetric optimal truss design, we usually assign a symmetric truss configuration. However, if a nonlinear programming approach such as sequential quadratic programming is used, such solution cannot be obtained in general even if it exists. Even for the case where no asymmetric solution exists and a symmetric initial solution is given, a conventional nonlinear programming approach does not converge to a symmetric solution due to accumulation of numerical error.

In practical situations, a symmetric solution can be obtained by assigning equality constraints on the variables, or by simply linking the design variables. It is not clear, however, whether the optimal cross-sectional areas for symmetric truss configuration is really symmetric, and whether additional constraints are not restricting the design space to exclude a possibility for obtaining an asymmetric optimal solution that has smaller objective value than any symmetric solution.

In this study, we discuss symmetry property of an optimal solution of the Semi-Definite Program (SDP) problem. In the authors' recent paper [1], we have observed for topology optimization problems of symmetric trusses that symmetric optimal solutions are obtained without any additional constraints if we use a primal-dual interior-point method for SDP. The theoretical background behind such phenomenon will be investigated.

SDP is a class of convex mathematical programming and has various fields of application [2], including structural optimization. The effectiveness of SDP has been shown for topology optimization of trusses under constraints on compliance [3] and the fundamental eigenvalue of free vibration [1]. The SDP can be solved in polynomial time worst-case complexity by using the primal-dual interior-point method which has been first developed for linear program [4], and has been successfully extended to SDP [5,6]. In many interior-point methods, an optimal solution can be obtained by numerically tracing the interior path which is referred to as central path. It is very important, therefore, to investigate the properties of the central path of SDP. Although several studies have been presented concerning the properties of the central path [7,8], no study has been reported for the SDP problem with certain symmetry. In this study, we will prove that a symmetric optimal solution is theoretically guaranteed to be obtained for a *symmetric SDP* if a primal-dual interior-point method is used.

4. Primal-dual pair of SDP and central path

Let S^n denote the sets of all $n \times n$ real symmetric matrices. $S^n_+ \subset S^n$ and $S^n_{++} \subset S^n_+$ denote the sets of all positive semidefinite and positive definite real symmetric matrices, respectively. The notation $\boldsymbol{U} \bullet \boldsymbol{V}$ is used to stand for the inner product of $\boldsymbol{U}$ and $\boldsymbol{V} \in \mathrm{R}^{n \times n}$:

$$\boldsymbol{U} \bullet \boldsymbol{V} = \sum_{i=1}^{n} \sum_{j=1}^{n} U_{i,j} V_{i,j} .$$

Let $\boldsymbol{F}_i \in S^n \quad (i = 1,2,\cdots m)$, $\boldsymbol{b} \in \mathrm{R}^m$ and $\boldsymbol{C} \in S^n$. The standard form SDP problem and its dual are formulated as

$$\begin{aligned} \mathrm{P:} \quad & \min \quad \boldsymbol{C} \bullet \boldsymbol{X} \\ & \mathrm{s.t.} \quad \boldsymbol{F}_i \bullet \boldsymbol{X} = b_i, \ (i = 1,2,\cdots,m), \ \ \boldsymbol{X} \in S^n_+ ; \end{aligned}$$

D: $\max \sum_{i=1}^{m} b_i y_i$

s.t. $\sum_{i=1}^{m} \boldsymbol{F}_i y_i + \boldsymbol{Z} = \boldsymbol{C}$, $\boldsymbol{Z} \in S_+^n$.

Here, $\boldsymbol{X}$ and $\boldsymbol{Z}$ are variable matrices and $\boldsymbol{y} = (y_i) \in \mathrm{R}^m$ is a variable vector. Throughout the paper, the linear independence of $\boldsymbol{F}_i$ $(i = 1,2,\cdots m)$ and the existence of a feasible solution $(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{Z})$ satisfying $\boldsymbol{X} \in S_{++}^n$ and $\boldsymbol{Z} \in S_{++}^n$ are assumed.

The central path for the SDP problems (P and D) is a trajectory of the solutions $(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{Z})$ to a family of the following parametric problem (CP_μ) with respect to the parameter $\mu > 0$:

$$(\mathrm{CP}_\mu) \qquad \boldsymbol{XZ} = \mu \boldsymbol{I}, \qquad (\mu > 0), \tag{1}$$

$$\boldsymbol{F}_i \bullet \boldsymbol{X} = b_i, \qquad (i = 1,2,\cdots,m), \tag{2}$$

$$\sum_{i=1}^{m} \boldsymbol{F}_i y_i + \boldsymbol{Z} = \boldsymbol{C}, \tag{3}$$

$$\boldsymbol{X}, \boldsymbol{Z} \in S_{++}^n, \tag{4}$$

where $\boldsymbol{I}$ denotes the $n \times n$ identity matrix. Letting $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu))$ denote the solution to (CP_μ), the following theorem has been obtained:

Theorem 4.1. ([6] Theorem 3.1.) (CP_μ) *has a unique solution* $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu))$.

Notice here that $(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{Z})$ is the optimal solution to SDP (P and D) if and only if it satisfies Eqs.(1)-(4) with $\mu = 0$. The primal-dual interior-point method [2,6] computes a solution, which is written as $(\overline{\boldsymbol{X}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{Z}})$, by tracing the central path as $\mu \to 0$. Therefore, the obtained $(\overline{\boldsymbol{X}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{Z}})$ can be regarded as the limit $\mu \to 0$ of $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu))$. In order to show the symmetry property of $(\overline{\boldsymbol{X}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{Z}})$, we investigate the symmetry property of $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu))$ in the following.

5. Definitions and properties of operators

Let $\Pi_n = \{\Pi_n(i) \mid i = 1,2,\cdots,n\}$ denote a permutation of n indices $1,2,\cdots,n$, where $\Pi_n(i)$ stands for the location of index i in Π_n. $\boldsymbol{e} = (e_i) \in \mathrm{R}^n$ denotes the vector satisfying $e_i = 1$ or -1 $(1,2,\cdots,n)$. The operators $S(\Pi_n) : \mathrm{R}^n \mapsto \mathrm{R}^n$ and $Q(\Pi_n, \boldsymbol{e}) : \mathrm{R}^{n \times n} \mapsto \mathrm{R}^{n \times n}$ are defined for Π_n and $\boldsymbol{e}$ as

Definition 5.1. *The operators* $S(\Pi_n)$ *and* $Q(\Pi_n, \boldsymbol{e})$ *are defined for a vector* $\boldsymbol{p} = (p_i) \in \mathrm{R}^n$ *and* $\boldsymbol{A} = [A_{i,j}] \in \mathrm{R}^{n \times n}$*, respectively, such that applications of* $S(\Pi_n)$ *and* $Q(\Pi_n, \boldsymbol{e})$ *result in* $\boldsymbol{p}^{S(\Pi_n)}$ *and* $\boldsymbol{A}^{Q(\Pi_n, \boldsymbol{e})}$ *satisfying*

$$p_i^{S(\Pi_n)} = P_{\Pi_n(i)}, \quad A_{i,j}^{Q(\Pi_n, \boldsymbol{e})} = A_{\Pi_n(i), \Pi_n(j)} e_i e_j, \quad (i, j = 1,2,\cdots,n).$$

In the subsequent discussions, $S(\Pi_n)$ and $Q(\Pi_n, \boldsymbol{e})$ are often abbreviated by S and Q, respectively, if their dependence on Π_n and $\boldsymbol{e}$ is understood from the context. From Definition 5.1, the following properties for a permutation Π_n and a vector $\boldsymbol{e} \in \mathrm{R}^n$ are deduced immediately:

Property 5.2. *For matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B} \in \mathrm{R}^{n \times n}$,

$$(\boldsymbol{AB})^Q = \boldsymbol{A}^Q \boldsymbol{B}^Q, \tag{5}$$

$$\boldsymbol{A}^Q \bullet \boldsymbol{B}^Q = \boldsymbol{A} \bullet \boldsymbol{B}, \tag{6}$$

$$((\boldsymbol{A})^Q)^Q = \boldsymbol{A}. \tag{7}$$

Property 5.3. *For a matrix* $\boldsymbol{D} \in S^n$,

$$\boldsymbol{D} \in S_{++}^n \Leftrightarrow \boldsymbol{D}^Q \in S_{++}^n. \tag{8}$$

6. Symmetric SDP and symmetry of its solution

6.1 Symmetry with respect to permutations

Consider the following symmetry conditions for matrices $\boldsymbol{F}_i$ and $\boldsymbol{C}$, and a vector $\boldsymbol{b}$ of SDP problem (P and D).

Condition 6.1. *There exist permutations* Π_m, Π_n *and a vector* $\boldsymbol{e} \in \mathrm{R}^n$ *such that*

$$\boldsymbol{b}^{S(\Pi_m)} = \boldsymbol{b}\,, \tag{9}$$

$$\boldsymbol{C}^{Q(\Pi_n,\boldsymbol{e})} = \boldsymbol{C}\,, \tag{10}$$

$$\boldsymbol{F}_i^{Q(\Pi_n,\boldsymbol{e})} = \boldsymbol{F}_{\Pi_m(i)}\,, (i = 1,2,\cdots,m)\,. \tag{11}$$

The SDP that satisfies Conditon 6.1 is referred to as a *symmetric SDP*.

In the following discussions, we fix the parameter μ to μ^* and the solution to (CP_{μ^*}) (Eqs.(1)-(4)) is denoted by $(\boldsymbol{X}^*,\boldsymbol{y}^*,\boldsymbol{Z}^*) = (\boldsymbol{X}(\mu^*),\boldsymbol{y}(\mu^*),\boldsymbol{Z}(\mu^*))$. The following theorem guarantees the symmetry of the solution on the central path of a symmetric SDP.

Theorem 6.2. *Suppose* $\boldsymbol{b}$, $\boldsymbol{F}_i$ *and* $\boldsymbol{C}$ *satisfy Conditon 6.1. Then* $(\boldsymbol{X}^*,\boldsymbol{y}^*,\boldsymbol{Z}^*)$ *satisfies*

$$\boldsymbol{X}^{*Q} = \boldsymbol{X}^*,\ \ \boldsymbol{y}^{*S} = \boldsymbol{y}^*,\ \ \boldsymbol{Z}^{*Q} = \boldsymbol{Z}^*. \tag{12}$$

Proof. Since $(\boldsymbol{X}^*,\boldsymbol{Z}^*)$ satisfies Eq.(4), it follows from Eq.(8) that $\boldsymbol{X}^{*Q}$ and $\boldsymbol{Z}^{*Q} \in S^n_{++}$. We obtain

$$\begin{aligned}
&\boldsymbol{F}_i \bullet \boldsymbol{X}^{*Q} \\
&= \boldsymbol{F}_i^Q \bullet \boldsymbol{X}^* && \text{(from Eq.(6) and Eq.(7))} \\
&= \boldsymbol{F}_{\Pi_m(i)}^Q \bullet \boldsymbol{X}^* && \text{(from Eq.(11))} \\
&= b_{\Pi_m(i)} && \text{(from Eq.(2))} \\
&= b_i, && \text{(from Eq.(9))}
\end{aligned}$$

which implies that $\boldsymbol{X}^{*Q}$ satisfies Eq.(2). It can be obtained that

$$\begin{aligned}
&\sum_{i=1}^{m} \boldsymbol{F}_i y^*_{\Pi_m(i)} + \boldsymbol{Z}^{*Q} \\
&= \sum_{i=1}^{m} \boldsymbol{F}_i^Q y_i^* + \boldsymbol{Z}^{*Q} && \text{(from Eq.(11))} \\
&= \left(\sum_{i=1}^{m} \boldsymbol{F}_i\ y_i^* + \boldsymbol{Z}^* \right)^Q \\
&= \boldsymbol{F}_0^Q && \text{(from Eq.(3))} \\
&= \boldsymbol{F}_0. && \text{(from Eq.(11))}
\end{aligned}$$

Thus, $(\boldsymbol{y}^{*S},\boldsymbol{Z}^{*Q})$ satisfies Eq.(3). $(\boldsymbol{X}^{*Q},\boldsymbol{Z}^{*Q})$ satisfies Eq.(1) because

$$\begin{aligned}
&\boldsymbol{X}^{*Q}\,\boldsymbol{Z}^{*Q} \\
&= (\boldsymbol{X}^*\boldsymbol{Z}^*)^Q && \text{(from Eq.(5))} \\
&= (\mu\boldsymbol{I})^Q && \text{(from Eq.(1))} \\
&= \mu\boldsymbol{I}\,.
\end{aligned}$$

Hence $(\boldsymbol{X}^{*Q},\boldsymbol{y}^{*S},\boldsymbol{Z}^{*Q})$ is also a solution of the problem (CP_{μ^*}), and uniqueness of the solution (Theorem 4.1) leads to Eq.(12).

6.2 Invariance of solution with respect to basis transformation

Let $\hat{\boldsymbol{F}}_i$ and $\hat{\boldsymbol{C}}$ be defined with an orthogonal matrix $\boldsymbol{H} \in \mathrm{R}^{n \times n}$ (i.e. $\boldsymbol{H}^T = \boldsymbol{H}^{-1}$) as

$$\hat{\boldsymbol{F}}_i = \boldsymbol{H}^T \boldsymbol{F}_i \boldsymbol{H}, \qquad (i = 1,2,\cdots,m) \tag{13}$$

$$\hat{\boldsymbol{C}} = \boldsymbol{H}^T \boldsymbol{C} \boldsymbol{H}. \tag{14}$$

Let $(\hat{\mathrm{CP}}_\mu)$ denote the system Eqs.(1)-(4) with $(\boldsymbol{F}_i, \boldsymbol{C}) = (\hat{\boldsymbol{F}}_i, \hat{\boldsymbol{C}})$ ..The following theorem implies if $\boldsymbol{y}^*$ is a solution to (CP_{μ^*}), then it is also a solution to $(\hat{\mathrm{CP}}_{\mu^*})$.

Theorem 6.3. *Let* $\hat{\boldsymbol{X}}^*$ *and* $\hat{\boldsymbol{Z}}^* \in S^n$ *be defined as*

$$\hat{\boldsymbol{X}}^* = \boldsymbol{H}^T \boldsymbol{X}^* \boldsymbol{H}, \;\; \hat{\boldsymbol{Z}}^* = \boldsymbol{H}^T \boldsymbol{Z}^* \boldsymbol{H}.$$

Then, $(\hat{\boldsymbol{X}}^*, \boldsymbol{y}^*, \hat{\boldsymbol{Z}}^*)$ *is the unique solution of* $(\hat{\mathrm{CP}}_{\mu^*})$.

Consider an SDP problem defined by $\boldsymbol{b}$, $\hat{\boldsymbol{F}}_i$ and $\hat{\boldsymbol{C}}$ satisfying Eqs.(13) and (14), where $\boldsymbol{b}$, $\boldsymbol{F}_i$ and $\boldsymbol{C}$ satisfy Condition 6.1. Then the central path of the SDP problem corresponding to $\boldsymbol{b}$, $\hat{\boldsymbol{F}}_i$ and $\hat{\boldsymbol{C}}$ is defined as the trajectory of the solutions to $(\hat{\mathrm{CP}}_\mu)$. If $\boldsymbol{y}^*$ is a solution on the central path (CP_{μ^*}) of the symmetric SDP defined by $\boldsymbol{b}$, $\boldsymbol{F}_i$ and $\boldsymbol{C}$, then Theorem 6.3 implies that $\boldsymbol{y}^*$ is also a solution to $(\hat{\mathrm{CP}}_{\mu^*})$. In other words, if an SDP satisfies Conditon 6.1, we can choose an arbitrary basis for matrices and define an alternative SDP problem. Then an optimal solution $\bar{\boldsymbol{y}}$ obtained by a primal-dual interior-point method does not depend on the choice of the basis.

7. Application: optimization of trusses under eigenvalue constraints

7.1 Problem formulation

A truss configuration is given with fixed locations of nodes and members. The values and locations of nonstructural masses are also given. The optimal cross-sectional areas are found under constraints on fundamental eigenvalue of free vibration. Let n and m denote, respectively, the number of freedom of displacements and number of members of the truss. The vector of member cross-sectional areas, which are design variables, is denoted by $\boldsymbol{y} = (y_i) \in \mathrm{R}^m$. Let $\boldsymbol{K} \in S^n$ and $\boldsymbol{M}_s \in S^n$ denote the stiffness matrix and the mass matrix due to the structural mass both of which are functions of $\boldsymbol{y}$. The mass matrix for nonstructural masses is denoted by $\boldsymbol{M}_0 \in S^n$.

Let Ω_r denote the eigenvalue of free vibration. The lower bound of the eigenvalues is denoted by $\overline{\Omega}$. The optimization problem for specified fundamental eigenvalue is formulated as

$$\begin{aligned} \text{OP:} \quad & \min \;\; \sum_{i=1}^{m} c_i y_i \\ & \text{s.t.} \;\; \Omega_r \geq \overline{\Omega}, \;\; y_i \geq y_i' \;\; (r = 1,2,\cdots,n; i = 1,2,\cdots,m), \end{aligned}$$

where $\boldsymbol{c} = (c_i) \in \mathrm{R}^m$ is the vector of member lengths and y_i' is the lower bound for y_i.

The authors [1] showed that the problem OP is equivalent to the following D' with constant matrices $\boldsymbol{K}_i$ and $\boldsymbol{M}_i \in S^n$ $(i = 1,2,\cdots,m)$:

$$\begin{aligned} \text{D'}: \quad & \max \;\; -\sum_{i=1}^{m} c_i y_i \\ & \text{s.t.} \;\; \sum_{i=1}^{m} (\overline{\Omega}\boldsymbol{M}_i - \boldsymbol{K}_i) y_i + \boldsymbol{Z} = -\overline{\Omega}\boldsymbol{M}_0, \\ & \qquad \boldsymbol{Z} \in S_+^n, \;\; y_i \geq y_i', \;\; (i = 1,2,\cdots,m). \end{aligned}$$

Notice here that D' corresponds to D with $b_i = -c_i$, $\boldsymbol{F}_i = \overline{\Omega}\boldsymbol{M}_i - \boldsymbol{K}_i$ and $\boldsymbol{C} = -\overline{\Omega}\boldsymbol{M}_0$.

7.2 Symmetry of optimal design

Symmetric configuration of a truss is defined by the following conditions:

1. There is an axis or a plane of symmetry.

2. The locations of nodes and members are symmetric.

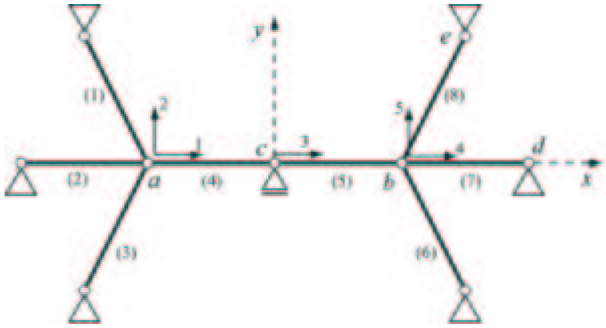

Fig. 1: 5 degree-of-freedom symmetric truss.

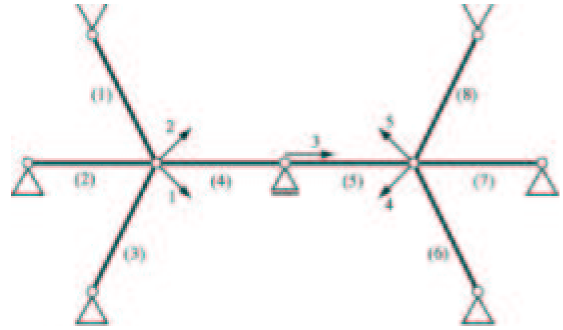

Fig. 2: Symmetric coordinate system.

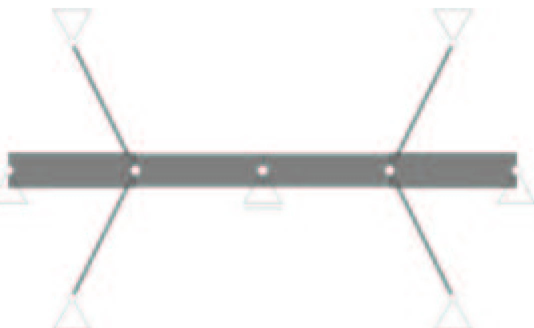

Fig. 3: Otimal solution obtained by SDPA.

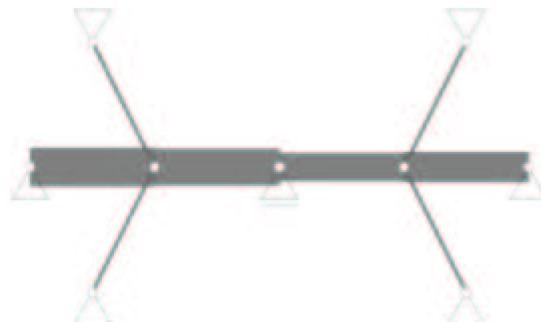

Fig. 4: An optimal solution obtained by SQP.

3. The locations and the values of the nonstructural masses are symmetric.
4. Support conditions are symmetric.

The vector of cross-sectional areas $\boldsymbol{y}$ is called a *symmetric design* if the symmetrically located members have the same cross-sectional areas. The term *symmetric truss* is used to stand for the truss with a symmetric configuration and a symmetric design. Since most of trusses actually built are symmetric, an optimal symmetric truss is desired to be obtained as a result of optimization. In such case, a symmetric truss configuration is given for the optimization problem.

Our concern is whether an optimal symmetric design is always obtained for a symmetric configuration.

For an example, consider a symmetric truss configuration and assignment of member number as shown in Fig.1. We can easily see that the vector of member lengths $\boldsymbol{c}$ satisfies $\boldsymbol{c}^{S(\Pi_m)} = \boldsymbol{c}$ for an appropriate Π_m; e.g. $\Pi_m = 8\,7\,6\,5\,4\,3\,2\,1$. Therefore, the corresponding SDP problem satisfies $\boldsymbol{b}^{S(\Pi_m)} = \boldsymbol{b}$ in Condition 6.1, where $b_i = -c_i$. Suppose the coordinate system such that the displacement of a node in a direction of the local coordinate, or the displacements of nodes of the same value in the directions of the local coordinates lead to a symmetric or an antisymmetric deformation. We refer to such a coordinate system as a *symmetric coordinate system.* An example of symmetric coordinate system is as shown in Fig.2. Then, for a symmetric coordinate system, it is straightforward to show that $\boldsymbol{K}_i^{Q(\Pi_n,\boldsymbol{e})} = \boldsymbol{K}_{\Pi_m(i)}$, $\boldsymbol{M}_i^{Q(\Pi_n,\boldsymbol{e})} = \boldsymbol{M}_{\Pi_m(i)}$ and $\boldsymbol{M}_0^{Q(\Pi_n,\boldsymbol{e})} = \boldsymbol{M}_0$ for appropriate Π_m, Π_n and $\boldsymbol{e}$, which implies that $\boldsymbol{F}_i = \overline{\Omega}\boldsymbol{M}_i - \boldsymbol{K}_i$ and $\boldsymbol{C} = -\overline{\Omega}\boldsymbol{M}_0$ satisfy Condition 6.1. Usually the coordinates are assigned in the same directions as one of the global coordinates as shown in Fig.1 and, generally, an SDP problem formulated by using such a coordinate system does not satisfy Condition 6.1. However, the relation of the symmetric coordinate system and any orthogonal coordinate system is written by the transformation matrix $\boldsymbol{H}$. Then Theorem 6.3 implies that a symmetric optimal design $\boldsymbol{y}$ is guaranteed to be obtained for any orthogonal coordinate system and assignment of member and coordinate numbers.

Note that a space truss sometimes has more than one plane of symmetry. In such case, we simply apply Π_m, Π_n, $\boldsymbol{e}$ and $\boldsymbol{H}$ successively for each plane, and we can prove symmetry of the solution with respect to each of the selected plane.

7.3. Examples

Optimal cross-sectional areas are found for symmetric plane trusses by using the SDP software; SDPA [9]. Sequential Quadratic Programming (SQP) [10] is also applied for comparison purpose. The material of the members is steel where Young's modulus and density are 205.8 Gpa and 7.86×10^{-3} kg/cm^3. The specified eigenvalue is 1000.0 rad^2/s^2, and $y' = 10.0\,\text{cm}^2$ for all the members.

Consider a five-degree-of-freedom plane truss configuration as shown in Fig.1. Nonstructural masses are 2.1×10^5 kg at nodes a and b, and 2.1×10^6 kg at node c. This configuration is symmetric with respect to the x- and y-axes, and $m=8$, $n=5$. The nodal coordinates of nodes b, c, d and e are assigned as (100.0 cm, 0.0), (0.0, 0.0), (200.0 cm, 0.0) and (150.0 cm, 150.0 cm), respectively.
Note that $\boldsymbol{F}_i$ $(i=1,2,\cdots,8)$ are linearly independent for this configuration. Consider a symmetric coordinate system as shown in Fig.2. From the symmetry properties with respect to x- and y-axes, it can be seen that $\boldsymbol{b}$, $\boldsymbol{F}_i$ and $\boldsymbol{C}$ with respect to the symmetric coordinate system satisfy Condition 6.1 with $\Pi_m = 8\,7\,6\,5\,4\,3\,2\,1$, $\Pi_n = 4\,5\,3\,1\,2$ and $\boldsymbol{e}=(1,1,-1,1,1)$, or with $\Pi_m = 3\,2\,1\,4\,5\,8\,7\,6$, $\Pi_n = 2\,1\,3\,5\,4$ and $\boldsymbol{e}=(1,1,1,1,1)$. The basis transformation matrix $\boldsymbol{H}$ from the symmetric coordinate system (Fig.2) to the coordinates as shown in Fig.1 can be obtained as

$$\boldsymbol{H}=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

If D' is formulated with respect to the coordinate system in Fig.1, then, from Theorem 6.3, symmetric optimal cross-sectional areas are guaranteed to be obtained without any pre-process by using a primal-dual interior-point method.

9. Conclusions

It has been proved that the solution on the central path of a symmetric SDP is symmetric. Since an optimal solution is obtained as the limit of the central path, the optimal solution obtained by an interior-point method that traces the central path is also symmetric even if there may exist other asymmetric optimal solutions.
Optimization problem of a symmetric truss configuration for specified fundamental eigenvalue of vibration has been formulated as a symmetric SDP. It has been proved that a symmetric optimal truss design exists and can be obtained by a primal-dual interior-point method. It has been shown through numerical experiments that the symmetric solution can be obtained by an SDP software even for the case where the conventional nonlinear programming approach converges to an asymmetric optimal solution.

10. References

1. Ohsaki M, Fujisawa K, Katoh N and Kanno Y. Semi-definite programming for topology optimization of trusses under multiple eigenvalue constraints. Comput. Meth. Appl. Mech. Eng., 1999, 180:203-217
2. Vandenberghe L and Boyd S. Semidefinite programming. SIAM Review, 1996, 38:49-95
3. Ben-Tal A and Nemirovski A. Robust truss topology optimization via semidefinite programming. SIAM J. Optim., 1997, 7:991-1016
4. Megiddo N. Pathways to the optimal set in linear programming. In: Megiddo, N. (ed.) Progress in Mathematical Programming: interior-Point Algorithms and Related Methods. New York: Springer-Verlag, 1989, 131-158
5. Alizadeh F, Haeberly J-PA and Overton ML. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. SIAM J. Optim., 1998, 8:746-768
6. Kojima M, Shindoh S and Hara S. Interior-point methods for the monotone semidefinite linear complementarity problems. SIAM J. Optim., 1997, 7:86-125
7. Goldfarb D and Scheinberg K. Interior point trajectories in semidefinite programming. SIAM J. Optim., 1998, 8:871-886
8. Halická M. Analytical properties of the central path at boundary point in linear programming. Math. Programming, 1999, 84:335-355, 1999
9. Fujisawa K, Kojima M and Nakata K. SDPA (Semidefinite Programming Algorithm) User's Manual. Tech. Report B-308, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Japan, 1999
10. Arora JS and Tseng CH. IDESIGN User's Manual Ver. 3.5. Optimal Design Laboratory, The University of Iowa, 1987