Optimum Design of Finite Dimensional Systems with Coincident Critical Points

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1. Abstract

The author's methods of design sensitivity analysis and optimization of coincident nonlinear critical load factors are shown to be effective for a finite dimensional system with moderately large numbers of design variables and degrees of freedom of displacements. Optimum designs under constraints on linear and nonlinear buckling load factors are found for spherical trusses subjected to distributed loads and a concentrated load at the center, respectively. It is shown that an approximate optimum design with coincident nonlinear critical points is obtained by scaling the optimum designs under constraints on linear buckling load factors even for the case with significant prebuckling deformation.

2. Keywords: Coincident critical points, optimum design, spherical truss, nonlinear buckling load factor

3. Introduction

In the design process of dome structures that resist to external loads mainly with in-plane stress and deformation, it is important to assign appropriate stiffness distribution so that the structure has enough safety considering instability against possible large design loads. For plates and column-type structures such as transmission towers and high-rise buildings, linear eigenvalue formulation is usually applied to evaluate buckling loads because the deformation before buckling is negligible. For a shallow shell subjected to loads in the normal direction of the surface, however, the effect of prebuckling deformation should be incorporated in evaluating the buckling loads.

Optimization of structures for specified linear buckling load factor has been extensively investigated including the case where the optimum design has multiple or repeated eigenvalues [1]. Recently, there have also been several studies for optimum design of structures for specified nonlinear buckling load factor considering prebuckling deformation [2,3]. Ohsaki *et al.* [4] presented an optimization method considering reduction of maximum load factor of a imperfection sensitive structure.

It is easy to find optimum designs for specified limit point load factor by using a gradient based mathematical programming approach, because the sensitivity coefficients of the limit point load factor with respect to the design variables such as the cross-sectional areas and nodal coordinates of finite dimensional systems are bounded. The sensitivity coefficients of bifurcation load factor, however, are generally unbounded for modification of design variables. For a symmetric structure subjected to symmetric proportional loads, a symmetric design modification is classified as a minor imperfection [5] or second order imperfection where the sensitivity coefficients are bounded even for a bifurcation point [6].

It is well known that an optimal solution under buckling constraints often has multiple critical load factors [7]. For such cases, it is very difficult to obtain optimal solution even for linear buckling load constraints. The method in Ref. 4 for nonlinear buckling, however, is applicable only for a solution with simple critical load factor. In the field of nonlinear stability analysis, the critical point with multiple null eigenvalues is called coincident critical point [8,9]. Recently, the first author has developed sensitivity analysis and optimization methods for problems with coincident critical load factors, and presented some solutions of a small truss [10].

In this paper, the method of design sensitivity analysis of coincident nonlinear critical loads and the formulation of optimum design under nonlinear buckling constraints presented in Ref. 10 are shown to be effective for a finite dimensional system with moderately large numbers of design variables and degree of freedom of displacements. Optimum designs with coincident critical points are found for a spherical truss. It is shown that imperfection sensitivity is not enhanced as a result of optimization, and the reduction of maximum load due to a minor imperfection is sometimes equivalent with that to a major imperfection.

4. Nonlinear Stability Analysis

Consider a finite dimensional elastic system subjected to a set of proportional loads $\mathbf{P} = \Lambda \mathbf{P}_0$, where Λ is the load factor and $\mathbf{P}_0$ is the constant vector of load pattern. The effect of prebuckling deformation should be considered in evaluating the buckling loads of shell-type structures subjected to loads in the normal direction of the surface. In this case, the equilibrium path should be traced by an incremental path-following analysis.

The vector of state variables such as nodal displacements is denoted by $\mathbf{Q} = \{Q_i\}$. Lagrangian formulation is used for defining the strains. The total potential energy $\Pi(\mathbf{Q},\Lambda)$ is a function of $\mathbf{Q}$ and Λ. Let S_i denote partial differentiation of Π with respect to Q_i. Stationary condition of Π with respect to Q_i leads to the following equilibrium equations:

$$S_i = 0\,,\quad (i = 1,2,\cdots,f) \tag{1}$$

where f is the number of degrees of freedom. Let t denote a parameter defining a point along the fundamental equilibrium path that originates the undeformed initial state. t may represent Λ, Q_i, or the arc-length of the path, and is written in general form as

$$t = g(\mathbf{Q}, \Lambda) \tag{2}$$

The Hessian of Π with respect to Q_i is denoted by $\mathbf{S} = [S_{ij}]$ which is called tangent stiffness matrix or stability matrix. The r th eigenvalue $\lambda_r(t)$ and eigenvector $\Phi_r(t) = \{\Phi_{ri}(t)\}$ of $\mathbf{S}(t)$ along the fundamental equilibrium path are defined by

$$\sum_{j=1}^{f} S_{ij}\Phi_{rj} = \lambda_r \Phi_{ri}, \quad (i = 1, 2, \cdots, f) \tag{3}$$

where Φ_r is normalized by

$$\sum_{j=1}^{f} (\Phi_{rj})^2 = 1, \quad (i = 1, 2, \cdots, f) \tag{4}$$

Note that the eigenvalues λ_r are numbered in increasing order; i.e. λ_1 is the lowest eigenvalue.
The value of Λ that satisfies $\lambda_1 = 0$ is called critical load factor which is denoted by Λ^{c}. The point satisfying $\lambda_1 = 0$ along the equilibrium path is called critical point which is indicated by $t = t^{\mathrm{c}}$. In the following, the values corresponding to $t = t^{\mathrm{c}}$ is denoted by a superscript $(\,)^{\mathrm{c}}$. It is well known that an optimum design under constraints of nonlinear buckling load factors often has a coincident critical point where more than one eigenvalue simultaneously vanish.

6. Design Sensitivity Analysis

Consider a finite dimensional structure defined by a design variable vector $\mathbf{A} = \{A_i\}$. A_i may directly correspond to the cross-sectional area or thickness of a structural element, or may be a scaling factor for a set of cross-sectional areas. Let a prime denote total differentiation with respect to A_i. Partial differentiation with respect to A_i while $\mathbf{Q}$ and Λ are fixed is denoted with a bar. Sensitivity analysis with respect to the design variable A_i is called *design sensitivity analysis* in the following, although imperfection sensitivity and design sensitivity are mathematically equivalent.
We consider a design modification that satisfies

$$\sum_{i=1}^{f} \bar{S}_i{}'\Phi_{1i}^{\mathrm{c}} = 0 \tag{5}$$

which is classified as a minor imperfection [6]. Eq. (4) indicates that the imperfection does not have direct effect on the internal force S_i in the direction of the buckling mode $\ddot{\mathbf{O}}_1^{\mathrm{c}}$. It is well known that the imperfection sensitivity coefficient of a bifurcation load factor corresponding to a major imperfection that does not satisfy Eq. (4) is not bounded. For a minor imperfection, however, the sensitivity coefficients are bounded even for a bifurcation point [6].
Suppose the design variables are modified while the lowest eigenvalue λ_1 of $\mathbf{S}$ is fixed at λ_1^{*}. The value of t where $\lambda_1 = \lambda_1^{*}$ is satisfied is denoted by t^{*}. Then t^{*} is a function of A_i. Other variables corresponding to the equilibrium state with $\lambda_1 = \lambda_1^{*}$ are also indicated by $(\,)^{*}$.
By differentiating Eqs. (1)-(4) with respect to A_i, and by using $\lambda_1{}' = 0$, the following equations are derived at $t = t^{*}$:

$$\sum_{j=1}^{f} S_{ij}Q_j^{*}{}' + \bar{S}_i{}' + S_{i\Lambda}\Lambda^{*}{}' = 0 \tag{6}$$

$$t^{*}{}' = \sum_{i=1}^{f} g_i Q_i^{*}{}' + g_\Lambda \Lambda^{*}{}' \tag{7}$$

$$\sum_{j=1}^{f}\sum_{k=1}^{f} S_{ijk} Q_k^{*}{}'\Phi_{1j} + \sum_{j=1}^{f} \bar{S}_{ij}\Phi_{1j} + \sum_{j=1}^{f} S_{ij\Lambda}\Phi_{1j}\Lambda^{*}{}' + \sum_{j=1}^{f} S_{ij}\Phi_{1j}^{*}{}' = \lambda_1\Phi_{1j}^{*}{}' \tag{8}$$

$$\sum_{j=1}^{f} \Phi_{1i}^{*}\Phi_{1i}^{*}{}' = 1 \tag{9}$$

where $(\,)_\Lambda$ denotes partial differentiation with respect to Λ, and g_i is the partial differentiation of g with respect to Q_i. The design sensitivity coefficients of $\mathbf{Q}^{*}$, Λ^{*}, t^{*} and Φ_{1i}^{*} for fixed value of λ_1 are found by solving a set of $2f+2$ linear equations Eqs. (6)-(9).

At the critical point with $\lambda_1 = 0$, those sensitivity coefficients cannot be obtained because the matrix for Eqs. (6)-(9) are singular [4]. Consider a process of tracing the equilibrium path by a finite increment Δt of the path parameter. Let t_b denote the value of t where $\lambda_1 < 0$ is first satisfied as t is increased, and define $\lambda_{1\mathrm{b}} = \lambda_1(t_\mathrm{b})$, $t_\mathrm{a} = t_\mathrm{b} - \Delta t$, $\lambda_{1\mathrm{a}} = \lambda_1(t_\mathrm{a})$. The sensitivity coefficients of t_a and t_b for fixed value of λ_1 at $\lambda_{1\mathrm{a}}$ and $\lambda_{1\mathrm{b}}$, respectively, are calculated from Eqs. (6)-(9). Then the sensitivity coefficients at $t = t^\mathrm{c}$ where $\lambda_1 = 0$ is satisfied are obtained by interpolation [5]; e.g.

$$\Lambda^\mathrm{c}{}' = \frac{\lambda_{1\mathrm{a}}\Lambda_\mathrm{b}{}' - \lambda_{1\mathrm{b}}\Lambda_\mathrm{a}{}'}{\lambda_{1\mathrm{a}} - \lambda_{1\mathrm{b}}} \tag{10}$$

where Λ_a and Λ_b are the load factors at $t = t_\mathrm{a}$ and t_b, respectively.

7. Optimization Problem for Specified Nonlinear Critical Load Factors

Let $V = (\mathbf{A})$ denote the total structural volume which is a function of the vector $\mathbf{A}$ of the design variables. The jth critical load factor along the fundamental equilibrium path is denoted by Λ^{cj}. The specified buckling load factor is denoted by $\overline{\Lambda}$. Then the optimization problem is formulated as [10]

$$\text{Minimize} \quad V = (\mathbf{A}) \tag{11}$$

$$\text{subject to} \quad \Lambda^{cj} \geq \overline{\Lambda}, \quad (j = 1,2,\cdots,s) \tag{12}$$

$$\lambda_j(\overline{\Lambda}) \geq 0, \quad (j = s+1, s+2, \cdots, q) \tag{13}$$

$$A_i \geq \overline{A}_i, \quad (i = 1,2,\cdots,m) \tag{14}$$

where s and q are the actual multiplicity of the buckling load factor and its possible upper bound. $\overline{A}_i$ is the minimum cross-sectional area, and m is the number of members. It has been observed in the numerical experiments that the value of s does not often change during the optimization process.

It is well known that the design sensitivity coefficients of the buckling load factor defined by a linear eigenvalue problem are discontinuous with respect to the design variable for the case of multiple eigenvalues [1]. Similar difficulties arise also for the case of coincident nonlinear critical points. Since λ_r $(r = 1,2,\cdots,s)$ are not exactly equal to 0 at $t = t_\mathrm{a}$ or t_b, the eigenvectors are defined distinctly, and the sensitivity coefficients are obtained from (6)-(9) without any difficulty. The eigenvalues corresponding to each eigenvectors, however, may intersect with each other in the region $t_\mathrm{a} \leq t \leq t_\mathrm{b}$. In this case, the correct pairs of eigenvalues and eigenvectors, which are detected by using the symmetry conditions and continuity, should be used at t_a and t_b in the interpolation equation (10).

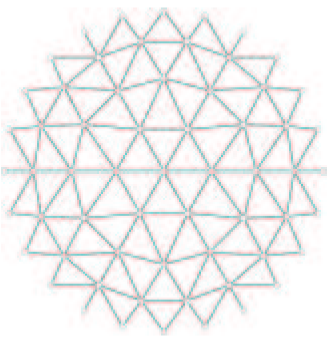
The sensitivity coefficients of λ_r that appear in (13) are obtained also by interpolation. Differentiation of (3) at $t = t_\mathrm{a}$ or t_b under the condition that λ_1 is fixed leads to

$$\sum_{j=1}^{f}\sum_{k=1}^{f} S_{ijk} Q_k^*{}' \Phi_{rj} + \sum_{j=1}^{f} \overline{S}_{ij} \Phi_{rj} + \sum_{j=1}^{f} S_{ij} \Phi_{rj}^*{}' = \lambda_r \Phi_{ri}^*{}' + \lambda_r^*{}' \Phi_{ri} \tag{15}$$

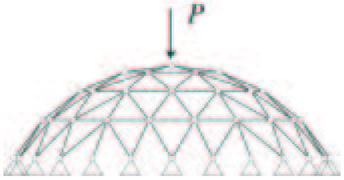
where $\lambda_r^*{}' \neq 0$ at this time for $r > s$. By premultiplying Φ_{ri} to (15), taking summation over i, and incorporating (2) and (3), the following equation is derived:

$$\lambda_r^*{}' = \sum_{i=1}^{f}\sum_{j=1}^{f}\sum_{k=1}^{f} S_{ijk} Q_k^*{}' \Phi_{ri} \Phi_{rj} + \sum_{i=1}^{f}\sum_{j=1}^{f} \overline{S}_{ij} \Phi_{ri} \Phi_{rj} \tag{16}$$

Note that $Q_i^*{}'$ has been already obtained from (6)-(9) at $t = t_\mathrm{a}$ and t_b, and $\Phi_{ri}^*{}'$ is not needed in evaluating $\lambda_r^*{}'$, which is similar to the case of linear eigenvalue problems. The sensitivity coefficients $\lambda_{\mathrm{a}r}{}'$ and $\lambda_{\mathrm{b}r}{}'$ are found by evaluating (16) at $t = t_\mathrm{a}$ and t_b, respectively. Then $\lambda_r^\mathrm{c}{}'$ at the critical point is calculated by the following interpolation equation:



(a) plan



(b) elevation

Fig. 1. A spherical truss.

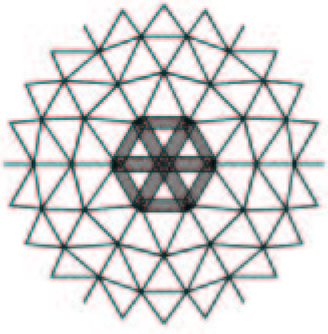


Fig. 2. Optimum design for concentrated load.

$$\lambda_r^{c}{}' = \frac{\lambda_{a1}\lambda_{br}{}' - \lambda_{b1}\lambda_{ar}{}'}{\lambda_{1a} - \lambda_{1b}} \tag{17}$$

Note that t is not fixed at t^{c} while evaluating $\lambda_r^{c}{}'$; i.e. t^{c} is also a function of A_i under the condition $\lambda_1 = 0$.

8. Examples

Consider a spherical truss as shown in Fig. 1, where $P = 9.8\,\mathrm{kN}$. The open angle is 120 deg. and the radius of the circle where the pin supports are located is 800 cm. The members in the meridian directions have the same length. The specified buckling load factor is 100. The elastic modulus is 205.8 GPa. $\overline{A}_i$ is equal to 1.0 cm^2 for nonlinear buckling constraints, and 0 for linear buckling constraints. The design variables are member cross-sectional areas. Based on the symmetry property, the members are divided into 14 groups of which members have the same cross-sectional area. The Lagrangian formulation is used for defining the strain-displacement relation. The fundamental equilibrium path is traced by using the displacement increment method. Optimum designs are found by DOT Ver. 5.0 [11], where the method of modified feasible direction is used.

The optimal cross-sectional areas are as shown in Fig. 2, where the width of each member is proportional to its cross-sectional area. It is observed from Fig. 2 that the members near th The total struct buckling load facto re 99.961, 99.95 d factors correspor point are close of buckling behavi

Let t ple, a limit point is increased from are found slightly Λ_3^{c} are a little

The l design is 228.153 nonlinear buckl ter node at buckling formation is very large and should be properly incorporated in evaluating the buckling loads of a single-layer spherical truss as shown in Fig. 1 subjected to a concentrated load. Variation of λ_r is plotted with respect to δ in Fig. 3. Note that the curve a in Fig. 3 corresponds to a limit-point-type mode. The curve b is duplicate and corresponds to bifurcation-type modes.

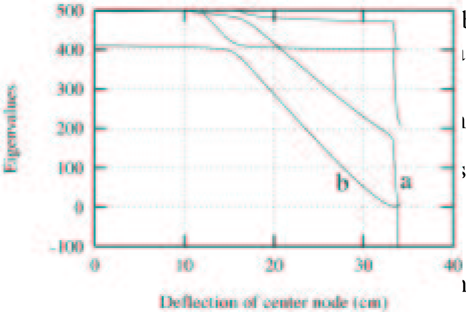


Fig. 3. Relation between deflection of center node and eigenvalues.

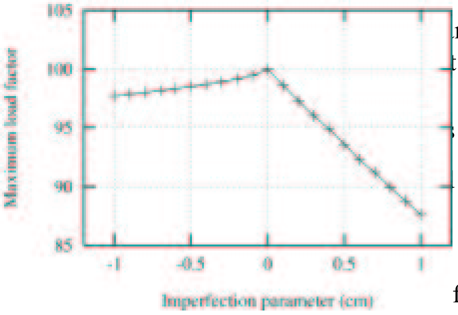
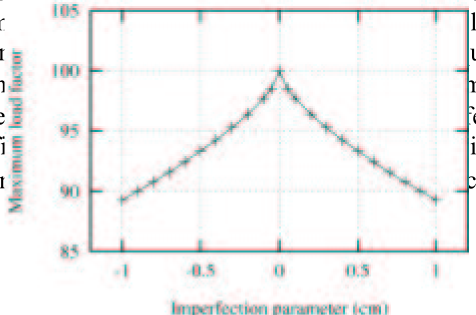
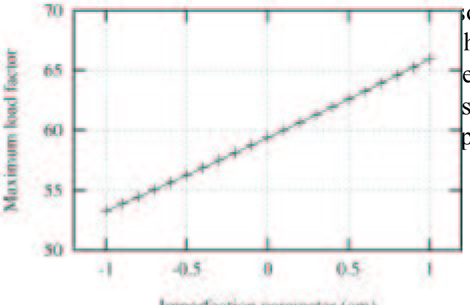


Fig. 4. Imperfection sensitivity in the direction of the limit-point-type mode.

In the following, imperfection characteristic of the optimal truss are investigated and compared with a non-optimal solution. Fig. 4 is the plot of maximum loads of imperfect systems. The nodes are dislocated in the direction of the symm limit-point-type i o that the maxir ual to 1. The horiz hich is the scalin mode. For $c > 0$, e negative z-dire fect system is a li sensitivity coeffi int is a bifurcatio ponds to a minor coefficients are al





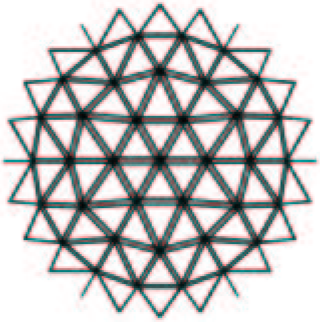


Fig. 7. Optimum design for distributed loads.

Fig. 5 shows the imperfection sensitivity in the direction of the antisymmetric buckling mode corresponding to the bifurcation-type instability. In this case, the imperfection corresponds to a major imperfection, and the imperfection coefficients are not bounded at $c = 0$. It should be noted, however, the magnitude of reduction of maximum load factor for this case is in the same order as symmetric imperfection in a finite range, e.g. $c = 1\,\mathrm{cm}$, of the imperfection parameter. Therefore, imperfection sensitivity at the perfect system corresponding to $c = 0$ is not important in practical situation, and minor imperfection should be properly considered in evaluating the maximum loads of imperfect systems. It has been confirmed that imperfection sensitivity characteristics in the directions of linear combination of the two bifurcation-type modes are similar to that in Fig. 5; i.e. no interaction of symmetric bifurcation points as the Augsti model [9] has been observed.

Fig. 6 shows the imperfection sensitivity in the direction of the limit-point-type mode of the initial design with $A_i = 20.0\mathrm{cm}^2$ for all the members. Since the critical point is a simple limit point, the maximum load factor is linear with respect to the imperfection parameter. The reduction of the maximum load, e.g., for $c = 1\,\mathrm{cm}$ is in the same order as that for the optimum design. Therefore the imperfection sensitivity does not increase as a result of optimization.

An optimum design has been also found under linear buckling constraints by using the semi-definite programming approach proposed in Ref. 12, which has also been effective for multiple frequency constraints [13]. The total structural volume is $2.75730\times10^5\ \mathrm{cm}^3$ and the nonlinear buckling load factor of the optimum design is 21.6916. Let A^{NL} and A^{LIN} denote the optimum cross-sectional areas, respectively, under nonlinear and linear buckling constraints. Note that A^{LIN} is almost proportional to A^{NL} which has been shown in Fig.1. If we scale A^{LIN} to satisfy $\Lambda_1^{\mathrm{c}} = 100$, then the total volume is $1.27224\times10^6\ \mathrm{cm}^3$ which is only slightly more than that of A^{NL} . Therefore, for this case, it is practically acceptable to obtain an optimum design under linear buckling constraint and just scale it up to have approximate optimum design under nonlinear buckling constraints.

The optimal cross-sectional areas for the case of uniformly distributed vertical loads is as shown in Fig. 7. It is observed from Fig. 7 that the cross-sectional areas are almost same except those of the members connected to the supports. In this case, six critical points are closely located. The linear buckling load factor of the optimum design is 109.866 which exceeds the specified nonlinear buckling load factor only about 10%. Therefore, prebuckling deformation is negligible for the case of uniformly distributed loads.

9. Conclusions

The author's method of design sensitivity analysis of coincident nonlinear critical loads and the formulation of optimum design under nonlinear buckling constraints have been shown to be applicable to a finite dimensional system with moderately large degree of freedom of displacements. Optimum designs with closely spaced critical points have been found for a spherical truss, and imperfection sensitivity of optimumdesigns has been discussed in detail.

It has been shown that optimization does not have any disadvantage in view of reduction of maximum load factor.

For spherical trusses, the buckling modes do not interact strongly with each other as a result of optimization, because the closely located critical points are a limit point and symmetric bifurcation points. It has also been sown that the magnitude of reduction of maximum load factor due to a symmetric imperfection that is classified as minor imperfection may be in the same order as that due to an antisymmetric major imperfection. Therefore, the fact that the imperfection sensitivity of the bifurcation load factor is unbounded is not important in practical situation. Minor imperfection should be properly considered in evaluating the maximum loads of imperfect systems.

The optimum designs have been compared with those under constraints of linear buckling load factors. It has been shown that the optimum designs under linear and nonlinear buckling constraints are almost same for the case of distributed loads. Although the effect of prebuckling deformation is very large for a spherical truss that carries a concentrated nodal load, an approximate optimum design may successfully be obtained by scaling the optimum design under constraints on linear buckling loads.

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