LARGE DEFLECTION ANALYSIS OF CABLE NETWORKS BY SECOND-ORDER CONE PROGRAM

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ABSTRACT
A method is presented for equilibrium shape analysis of cable networks considering geometrical and material nonlinearities. The Second-Order Cone Programming (SOCP) problem which has the same solution as that of the minimization problem of total potential energy is solved to obtain the equilibrium configuration. No assumption of stress state is needed in the proposed method. Therefore no process of trial-and-error is involved. Since SOCP can be solved by using the well-developed software based on the interior-point method, no effort is required to develop any analysis software. Numerical experiments are shown to demonstrate the efficiency of the proposed method.

1. INTRODUCTION
Various papers have been published concerning equilibrium shape analysis of cable networks considering geometrical nonlinearity [1–4]. Argyris and Scharpf [1] proposed the incremental method based on the tangent stiffness. Few studies, however, have explicitly dealt with the fact that cables are not capable of transmitting compressional forces, which is referred to as the stress-unilateral behavior. Panagiotopoulos [5] formulated variational inequality of cable networks for infinitesimal incremental displacements. The complementary energy principle for a cable modeled as one-dimensional continuum has been presented for large deflection analysis [6].

In the existing methods based on the tangent stiffness, the stress-unilateral behavior is modeled by absence of stiffness of slackening members. An assumption, however, is required whether each member will be in tensile or slackening state at the next step of increment. The formulation in [5] also should make assumptions on stress state to carry out the analysis. Since the assumed stress state may conflict with the obtained results from displacement increments, the trial-and-error process which is similar to that of elastoplastic analysis should be carried out at each incremental step. Such assumption and trial-and-error process, however, sometimes cause the divergence of solution. Another difficulty exists in the singularity of the tangent stiffness matrix in the case where many members are slackening.

Recently, there has been an extensive interest in methods and theories of nonlinear programming to apply to equilibrium analysis of structures [7] and to contact problems [8]. Mallett and Schmit [9] presented a method for equilibrium analysis of trusses where a nonlinear programming based on energy principle is solved. To authors' knowledge, all the studies except Cannarozzi [6] are based on small deflection or infinitesimal increments.

In this paper, a method is proposed for equilibrium shape analysis of cable networks based on Second-Order Cone Programming (SOCP) [10]. SOCP is a convex mathematical programming containing linear programming, convex quadratic programming, etc., and known to have several applications [11, 12].

The problem considered in this paper is non-linear because of the stress-unilateral behavior as well as large deflection. The material non-linearity is also considered. Our concern is to develop a method applicable even to the case where determination of stress states of cables is difficult, and/or to the case where the tangent stiffness matrix becomes singular at the initial, final, or intermediate states of analysis. For this purpose, an SOCP problem is formulated so as to have the same optimal solution as that of the minimization problem of total potential energy. Equilibrium shapes are obtained as the solution of the SOCP by using an interior-point method.

2. PROBLEM FORMULATIONS BASED ON SOCP

Consider a cable network in three dimensional space. We use the assumptions of small strain and large rotation. The network is discretized into members which connect pin joints and supports, and are not capable of transmitting compressional forces.

Let N^d denote the number of freedom of displacements. $\boldsymbol{x} \in \Re^{N^d}$ and $\bar{\boldsymbol{f}} \in \Re^{N^d}$ denote the vectors of coordinates of internal nodes that are not supported and the corresponding external dead loads. Our purpose is to find $\boldsymbol{x}$ at the equilibrium state for the specified nodal coordinates of the supports, external load $\bar{\boldsymbol{f}}$, initial unstressed length l_i^0 and cross-sectional area A_i of the ith member. Let c_i denote the elongation of the ith member. N^m denotes the number of members. The relation between c_i and $\boldsymbol{x}$ is written as

$$c_i = \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0, \quad (i = 1, 2, \cdots, N^m). \tag{1}$$

Here, $\boldsymbol{B}_i \in \Re^{3 \times N^d}$ and $\boldsymbol{b}_i^0 \in \Re^3$ are constant matrix and vector, respectively.

2.1. Linear elastic material

For a linear elastic material, the axial force N_i of the ith member is written as

$$N_i = \begin{cases} k_i c_i & (0 \leq c_i), \\ 0 & (c_i < 0), \end{cases} \tag{2}$$

where $k_i = EA_i / l_i^0$, and E is the elastic modulus. Then the strain energy $w_i = w_i(c_i)$ is defined by

$$w_i = \int_0^{c_i} N_i(c_i) dc_i. \tag{3}$$

From (1)–(3), the problem of minimum total potential energy is formulated as

$$\left.\begin{aligned} (\Pi):\quad & \text{Minimize} \quad \sum_{i=1}^{N^m} w_i - \bar{\boldsymbol{f}}^\top \boldsymbol{x} \\ & \text{subject to} \quad w_i = \begin{cases} \dfrac{1}{2} k_i c_i^2 & (0 \leq c_i), \\ 0 & (c_i < 0). \end{cases} \\ & \qquad c_i = \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0 \quad (i = 1, 2, \cdots, N^m). \end{aligned}\right\} \tag{4}$$

The solution $(\hat{\boldsymbol{x}}, \hat{w}_i, \hat{c}_i)$ of (Π) coincides with the nodal coordinates, the strain energy and member elongation at the equilibrium configuration, respectively.

Notice here that (Π) is a nonconvex optimization problem, and definition of w_i depends on the value of c_i. Therefore it is not recommended to solve (Π) by an algorithm of nonlinear programming. This motivates us to investigate the following (P) which is classified as an SOCP problem:

$$\left.\begin{aligned} (\mathrm{P}):\quad & \text{Minimize} \quad \Phi = \sum_{i=1}^{N^m} \frac{1}{2} k_i y_i^2 - \bar{\boldsymbol{f}}^\top \boldsymbol{x} \\ & \text{subject to} \quad y_i \geq \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - \bar{l}_i^0, \quad (i = 1, 2, \cdots, N^m). \end{aligned}\right\} \tag{5}$$

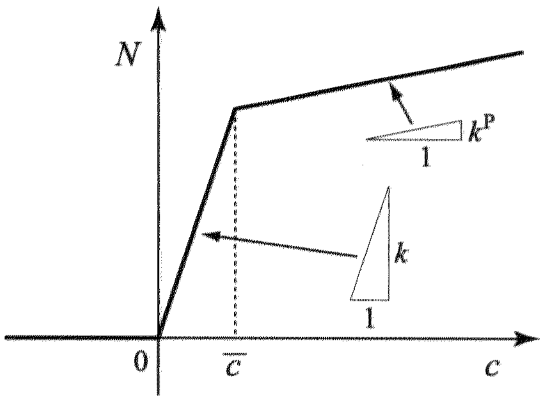


Figure 1: Bi-linear constitutive law.

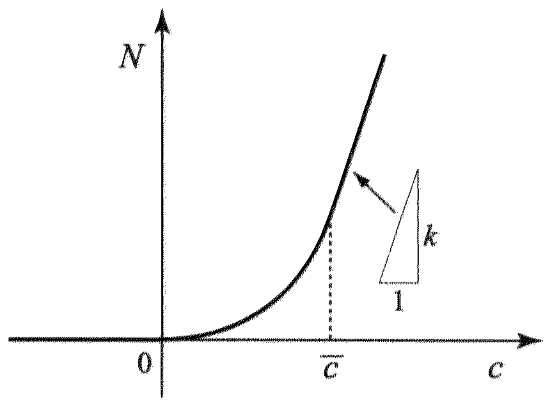


Figure 2: Non-linear constitutive law.

Let $(\boldsymbol{x}^*, y_i^*)$ denote a solution to (P). It follows from the convexity of Φ that $\boldsymbol{x}^*$ is also an optimizer of (Π). In this paper, we solve (P) to obtain the equilibrium shape.

2.2. Bi-linear material

Consider a bi-linear elastic material defined as

$$N_i = \begin{cases} k_i^p(c_i - \bar{c}_i) + k_i\bar{c}_i & (\bar{c}_i \leq c_i), \\ k_i c_i & (0 \leq c_i \leq \bar{c}_i), \\ 0 & (c_i < 0), \end{cases} \tag{6}$$

where $k_i^p \geq 0$. The relation between N_i and c_i is as illustrated in Fig. 1. Then the definition of w_i in (Π) should be replaced with

$$w_i = \begin{cases} \frac{1}{2}k_i^p(c_i - \bar{c}_i)^2 + k_i\bar{c}_i(c_i - \bar{c}_i) + \frac{1}{2}k_i\bar{c}_i^2 & (\bar{c}_i \leq c_i), \\ \frac{1}{2}k_i c_i^2 & (0 \leq c_i < \bar{c}_i), \\ 0 & (c_i < 0). \end{cases} \tag{7}$$

In this case, (Π) is equivalent to the following (P):

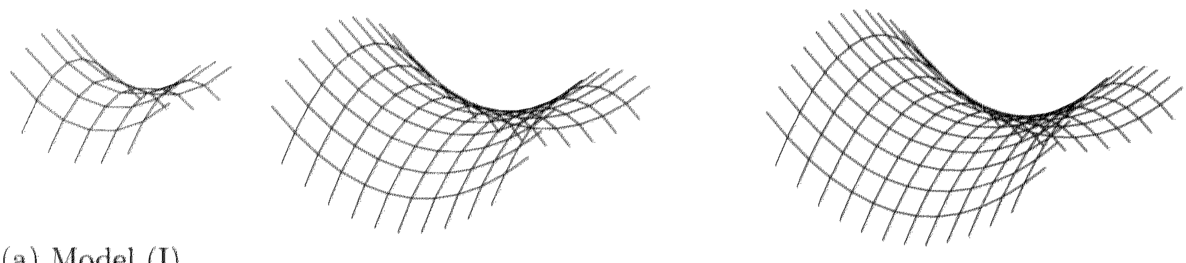
$$\left. \begin{array}{lll} \text{(P)}: & \text{Minimize} & \Phi = \sum_{i=1}^{N^m} \phi_i(y_{\alpha i}, y_{\beta i}) - \bar{\boldsymbol{f}}^\top \boldsymbol{x} \\ & \text{subject to} & (y_{\alpha i}, y_{\beta i}) \in C_i, \\ & & y_{\alpha i} + y_{\beta i} \geq \|\boldsymbol{B}_i\boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0, \quad (i = 1, 2, \cdots, N^m). \end{array} \right\} \tag{8}$$

Here,

$$\phi_i = \frac{1}{2}k_i y_{\alpha i}^2 + \frac{1}{2}k_i^p y_{\beta i}^2 + k_i\bar{c}_i y_{\beta i},$$
$$C_i = \{(y_{\alpha i}, y_{\beta i}) |\ y_{\alpha i} \leq \bar{c}_i,\ 0 \leq y_{\beta i}\}.$$

Notice again that (P) is an SOCP problem which is convex and has no partition of variables as (7) in the problem definition.

The constitutive law (6) is often used for elastoplastic analysis. Although elastoplastic behavior of structures is path-dependent, the strain energy is a function of c_i under assumption such that no member is unloaded at any stage of loading history. In this case,



(a) Model (I) (b) Model (II) (c) Model (III)

Figure 3: Initial solution (A).

the equilibrium shape can be obtained as the solution to (P), and optimal value of $y_{\alpha i}$ and $y_{\beta i}$ coincide with elastic and plastic components of elongation, respectively.

2.3. Stiffness reduction under small elongation

When the axial force of a cable member is relatively small, the stiffness appears to be smaller than that determined from the elastic modulus of the material. This phenomenon may be caused by two characteristics of the cable member; i.e. (i) the strand cable becomes loose; (ii) deflection of cable member due to its own weight. The following quadratic relation is assumed for a small range of c_i as illustrated in Fig. 2:

$$N_i = \begin{cases} k_i(c_i - \bar{c}_i) + k_i^s \bar{c}_i^2 & (\bar{c}_i \leq c_i), \\ k_i^s c_i^2 & (0 \leq c_i \leq \bar{c}_i), \\ 0 & (c_i < 0), \end{cases} \tag{9}$$

where k_i^s is determined such that N_i is continuously differentiable at $c_i = \bar{c}_i$. Then the problem (P) for this case is defined by

$$\phi_i = \frac{1}{3} k_i^s y_{\alpha i}^3 + \frac{1}{2} k_i y_{\beta i}^2 + k_i^s \bar{c}_i^2 y_{\beta i},$$
$$C_i = \{(y_{\alpha i}, y_{\beta i}) |\ 0 \leq y_{\alpha i} \leq \bar{c}_i,\ 0 \leq y_{\beta i}\}.$$

In a method based on the tangent stiffness matrix, the nonlinear constitutive law such as (9) must be linearized at each step of analysis. The problem (P), however, can be solved by using the interior-point method without any expansion or approximation of (9).

3. EXAMPLES

Equilibrium shapes are computed for cable networks with various sizes. Computation has been carried out on COMPAQ Alpha (CPU 21264 500MHz with 1GB memory). Self-equilibrium shapes with $\boldsymbol{f} = \boldsymbol{0}$ will be found in all the examples except those of elastoplastic analysis in Section 3.4.

Consider three cable networks as shown in Fig. 3 which are referred to as initial configurations (A) of Models (I), (II) and (III). The elastic modulus is $E = 205.8$ GPa and cross-sectional area is 10 cm^2 for each member. Each cable network projected to the horizontal plane makes a grid with 1 m $\times$ 1 m squares. The (x, y)-axes are defined along the grid, where the case of Model (III) is as shown in Fig. 4. Then the z-coordinate of the nodes are given as $z = (x^2 - y^2)/\alpha$, where $\alpha = 6$, 11 and 13 (m), respectively, for Models (I), (II) and (III). The two ends of all the cables are supported. The initial unstressed length of each member is 99% of the member length of (A). Note that the configuration (A) is close to the self-equilibrium configuration of each model. All the members are in tension at both (A) and the equilibrium state. Initial solutions (B)

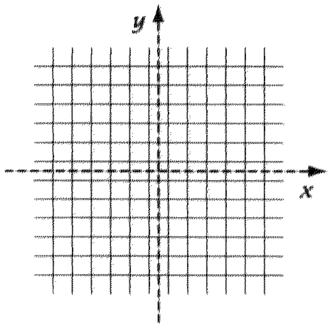


Figure 4: Coordinates (Model (III)).

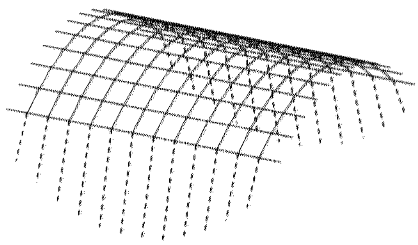


Figure 5: Initial solution (B) of Model (III).

Table 1: Comparison of performances of IPM (P), IPM (Π) and NR.

algorithm	model #	n	steps	CPU time (sec.) total	ave.	Φ ($\times 9.8$ kN $\cdot$ m)
IPM (P)	(I)	195	11	0.23	0.021	2.27560×10^3
	(II)	740	14	1.42	0.101	8.35266×10^3
	(III)	1056	15	2.46	0.164	1.18472×10^4
IPM (Π)	(I)	135	22	0.50	0.023	2.27560×10^3
	(II)	520	22	2.44	0.111	8.35266×10^3
	(III)	744	22	3.81	0.173	1.18472×10^4
NR	(I)	75	19	0.28	0.014	2.27560×10^3
	(II)	300	22	4.71	0.214	8.35266×10^3
	(III)	432	22	20.31	0.923	1.18472×10^4

are also defined with $z = ((\alpha/2)^2 - y^2)/\alpha$ for three models. The configuration (B) of Model (III) is as shown in Fig. 5, where only the dotted members are in tension.

Equilibrium shapes are computed by the following three methods:

IPM (P) : solve (P) by using the Interior-Point Method.
IPM (Π) : solve (Π) by using the Interior-Point Method.
NR : Newton-Raphson method.

NUOPT [13] is used for IPM (P) and IPM (Π), which is an implementation of the interior-point method for nonlinear programming. NR is the iterative method to update the nodal coordinates by using the tangent stiffness matrix at the current nodal coordinates and the residual nodal forces [1]. Notice here that IPM (P) and NR require the assumptions on the stress states of the members at each step of analysis.

3.1. Self-equilibrium shape analysis with the nonlinear constitutive law

Self-equilibrium shapes are found for Models (I)–(III) from initial configurations (A). All members obey the constitutive law (9) where $\bar{c}_i = 7.0 \times 10^{-4} l_i^0$.

The results by IPM (P), IPM (Π) and NR are as listed in Table 1. The obtained values of total potential energy Φ of three methods agree within 6 digits, which implies that these solutions satisfy equilibrium conditions. The number of independent variables for each method is denoted by n, where the formulation (P) requires the largest number of variables. Average CPU time means the computational time for each step.

It may be observed from Table 1 that the CPU time of IPM (P) is the smallest. The CPU time of IPM (P) and IPM (Π) is of order between n and n^2, whereas that of NR is of order between n^2 and n^3. IPM (P) is cheaper than IPM (Π) because (i) the process of IPM (Π) to investigate the stress state of each member requires additional CPU time at each step; (ii) the convexity of (P) causes the rapid convergence. Although the number of variables of (P) is more than that of (Π), it is shown that (P) has advantage over (Π).

Table 2: Comparison of performances between initial solutions (A) and (B).

algorithm	model #	n	steps (A)	steps (B)	CPU time (sec.) total (A)	ave.(A)	total (B)	ave.(B)	Φ ($\times 9.8$ kN $\cdot$ m)
IPM (P)	(I)	135	11	15	0.21	0.019	0.22	0.015	2.79779×10^3
	(II)	520	11	15	0.90	0.082	0.98	0.065	1.02694×10^4
	(III)	744	11	23	1.42	0.129	2.26	0.098	1.45659×10^4
IPM (Π)	(I)	135	11	15	0.24	0.022	0.23	0.015	2.79779×10^3
	(II)	520	11	15	0.93	0.082	1.05	0.070	1.02694×10^4
	(III)	744	15	31	2.10	0.140	3.28	0.106	1.45659×10^4
NR	(I)	75	22	*	0.26	0.012	*	*	2.79779×10^3
	(II)	300	20	*	4.24	0.212	*	*	1.02694×10^4
	(III)	432	9	*	7.40	0.820	*	*	1.45659×10^4

* algorithm cannot be applied.

Table 3: Comparison of results of Model (III').

algorithm	model constitutive law	n	steps	CPU time (sec.) total	ave.	Φ ($\times 9.8$ kN $\cdot$ m)
IPM (P)	(2)	744	45	5.37	0.120	3.11132×10^2
IPM (Π)	(2)	744	51	6.37	0.125	3.11132×10^2
NR	(2)	432	**	**	0.873	**
IPM (P)	(9)	1056	34	5.26	0.155	1.16644×10^2
IPM (Π)	(9)	744	**	**	0.978	**
NR	(9)	432	**	**	1.122	**

** did not converge within 1000 steps.

3.2. Comparison of performances from different initial solutions

Dependence of performances of three methods on initial solutions are investigated by comparing performances using the initial solutions (A) and (B). Suppose the constitutive law is given by (2).

The results are as listed in Table 2. For all the cases, it is observed that both IPM (P) and IPM (Π) converge to the equilibrium configurations, and IPM (P) has the best performance in view of computational cost. The computational cost for (B) is not very larger than that of (A), although (B) is much different from the equilibrium shape. Robustness of IPM (P) in view of initial solutions has been verified with this result. NR cannot be applied to (B) because the tangent stiffness matrix at (B) is singular.

3.3. Initial configuration with slackening members

Model (III') is defined so that the initial unstressed length of each member shown in blank line in Fig. 6 is equal to 102% of the length at the configuration (A) of model (III). Therefore, those members are slackening at the configuration (A). All the members are in tension at the equilibrium configuration. Although the configuration (A) contains slackening members, the tangent stiffness matrix is not singular at (A). The results using two different constitutive laws (2) and (9) are as listed in Table 3.

NR cannot find solutions for both cases because its trial-and-error process leads to oscillation of solutions. Note that the tangential stiffness matrix is not singular at each step of NR. IPM (Π) has not converged for the case of (9) while IPM (P) did not have any difficulty in both cases.

3.4. Elastoplastic analysis

Vertical loads of 4900 kN are applied at the center node of Model (I), and at the four nodes around the center of Models (II) and (III). The loaded nodes of Model (III) are as shown in Fig. 7 with the filled circles. Suppose the members obey elastoplastic constitutive law (6), where $\bar{c}_i = 0.03 l_i^0$ and $k_i^p = 0.1 k_i$.

Table 4: Comparison of results of elastoplastic analysis.

algorithm	model #	model n	steps	CPU time (sec.) total	CPU time (sec.) ave.	Φ ($\times 9.8$ kN $\cdot$ m)
IPM (P)	(I)	195	10	0.44	0.044	4.92416
	(II)	740	10	1.39	0.139	-2.09518×10^3
	(III)	1056	10	2.06	0.206	1.09141×10^2
IPM (II)	(I)	135	10	0.58	0.058	4.92416
	(II)	520	11	1.62	0.147	-2.09518×10^3
	(III)	774	11	2.17	0.197	1.09141×10^2
NR	(I)	75	25	0.32	0.013	4.92416
	(II)	300	32	6.92	0.194	-2.09518×10^3
	(III)	432	34	31.66	0.931	1.09141×10^2

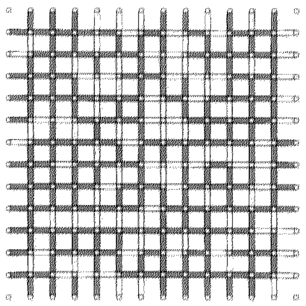


Figure 6: Slackening members of the initial solution (Model (III')).

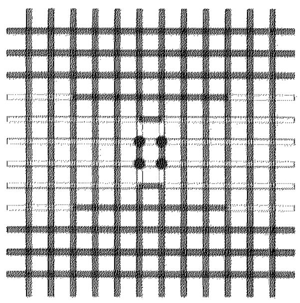


Figure 7: Plastic members.

The results by three methods are as listed in Table 4. For Model (III), the plastic members are as shown in blank lines in Fig. 7. It may be observed from Table 4 that the performance of IPM (P) is superior to those of the others in view of CPU time.

3.5. Unstable equilibrium shape analysis

The initial unstressed length of each member shown in blank line in Fig. 8 (a) is modified to 102% of the original length. These members are slackening at (A).

The solution obtained by IPM (P) is as shown in Fig. 8 (b). Note that the equilibrium configuration is not unique in this example. It can be seen from Fig. 8 (b) that four nodes around center are unstable. Therefore NR cannot find any solution because of the singularity of the tangent stiffness matrix. There exists, however, no difficulty in applying IPM (P) to find one of the equilibrium configurations.

4. CONCLUSIONS

An SOCP formulation which gives the same optimizer as that of the problem of minimum total potential energy for cable networks has been presented, and equilibrium shape has been obtained as a solution of the SOCP by using an interior-point method. In the proposed formulation, the material as well as the geometrical nonlinearity is considered. Since no assumption on stress state is included, the proposed algorithm does not involve any processes of trial-and-error.

It has been shown in the examples that the computational cost of the proposed method is small compared with those of simple minimization of the total potential energy and the Newton-Raphson method. Several examples have shown the effectiveness of this method for the cases of unstable structures or solutions with many members that has small axial forces, where the existing two methods have failed to obtain solutions.

(a) Initial solution. (b) Equilibrium state.

Figure 8: Slackening members.

In addition to these advantages, SOCP can be solved effectively by using well-developed softwares, therefore our task is only to input the geometrical and material properties of cable networks and no effort is required to develop any analysis softwares.

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