Topology and Geometry Optimization of Trusses and Frames

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1. INTRODUCTION

Early on in the design of structural systems, it is often desired to find the general layout of the system that most naturally and efficiently supports the anticipated design loads. This is sometimes done by optimizing the overall shape of the structural system as well as the connectivity (or topology) of structural elements comprising the structure. Variable topology shape optimization is much more general than fixed topology shape optimization in that it leads modifications to the connectedness (or "topology") of structural elements. It is performed early in the design of structural systems, the point at which there is the greatest amount of freedom, to find potentially optimal starting "concept designs" (Figure 1). There are two broad classes of techniques which can be applied to optimize the shape and topology of a structural system early on in the design process:

1. Discrete optimization of the structural system;
2. Continuum optimization of the structural system.

In the discrete optimization methods, the structure is generally modeled with discrete truss and/or beam/column elements, whereas in continuum methods, the structure is modeled as a continuum. Both methods have undergone extensive research and development in the past two decades, and both have their own strengths and weaknesses. In the following sections, both discrete and continuum structural shape/topology optimization methods are reviewed separately and then compared.

The history of shape and topology optimization of framed structures can be classified into three periods:

1. The initial period during which Michell (1904) and Maxwell (1894) made their pioneering studies in the field; Although the study by Michell is important in view of theoretical background, the techniques apply only to limited types of discrete structures and constraints (Hemp 1973). Following these initial works, the field of structural topology optimization fell relatively dormant for many decades.
2. A second period occured during the 1960's and 1970's in which time interest in structural optimization was re-kindled by the initial developments of high-speed computers. During this period, many important theoretical results for general optimization methods and numerical implementations were first presented, and difficulties in structural topology optimization were given extensive attention. In addition, methods for discrete shape/topology optimization were exercized on very small test problems due to computing limitations.
3. A third period during the 1980's and 1990's has been characterized by extremely dramatic growth in computing technologies. While theoretical work on structural topology optimization has continued, numerical techniques for structural topology optimization have been further refined, developed, and applied to larger-scale, more realistic structures. During this third period, while advances in discrete topology optimization continued, novel continuum structural topology optimization methods were also introduced and developed quite extensively.

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2. DISCRETE STRUCTURAL TOPOLOGY OPTIMIZATION

The initial design of skeletal structures such as trusses and frames can be broken down into the selection of nodal (or joint) locations, and design of the connectivity structural elements. The former process is referred to as *geometry optimization* or *configuration optimization*, and the latter is called *topology optimization*. Both processes sometimes include *sizing optimization* or traditional *optimum design* where the cross-sectional properties are optimized. *Simultaneous optimization of topology and geometry* may be referred to as *layout optimization*, or simply *configuration optimization*.

This review section on discrete structural topology optimization concentrates primarily on the numerical methods developed in the third period described above. The theoretical works such as explicit optimality criteria approach and the method based on so called *grillage* are summarized in the review papers by Kirsch (1989b) and books such as Rozvany (1976), Rozvany (1989) and Rozvany (1997). Since rigidly-jointed frames and pin-jointed trusses are modeled in a similar formulations of finite element method, trusses are mainly considered here. Optimizing topologies of trusses is more difficult than optimizing frames because of instability at the joints.

In the widely used numerical approach of topology optimization, unnecessary members are removed from highly connected *ground structure* while the nodal locations are fixed (Kirsch 1989b, Topping 1992). Many methods have been presented based on the ground structure approach, and finding practically useful solutions seems to be a matter of computational capacity. The main research topics in this field include the difficulties for the case with constraints on stresses, local buckling and frequency. Development of efficient computational algorithms for optimizing large scale trusses is also an important subject.

Geometry optimization of framed structures, on the other hand, seems to be rather straightforward. The nodal coordinates are considered as continuous design variable, and the optimal solutions may be found by using appropriate methods of mathematical programming. In this process, side constraints are usually given for the nodal coordinates to prevent existence of short members, or intersection of members. Therefore, topology cannot be optimized, and simultaneous optimization of topology and geometry is a very difficult problem that cannot be a simple extension of either topology optimization or geometry optimization.

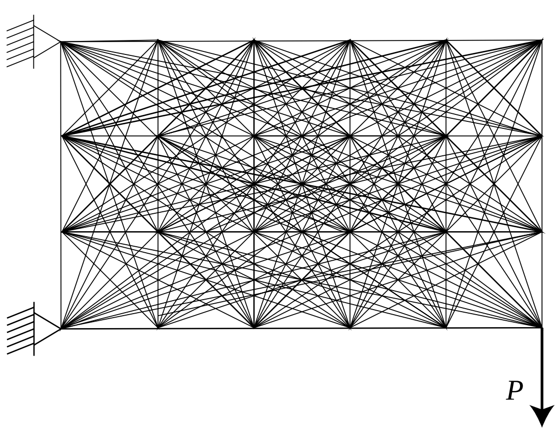


Figure 1. A highly connected ground structure.

2.1 Topology Optimization Problem

Consider a problem of finding optimal connectivity of the nodes from a set of existable nodes and members under constraints on responses for specified loading conditions. The structure with all the candidate nodes and members is called *ground structure*. A ground structure includes many nodes and members as illustrated in Figure 1. The approach that removes unnecessary members from the ground structure is called *ground structure approach*, and the paper by Dorn *et al.* (1964) is usually cited as the first work of this kind. There have been extensive researches based on this approach since 1960s (Dobbs and Felton 1969), and those are listed in the review articles by Kirsch (1989b), Topping (1984) and books such as Rozvany (1997).

Topology optimization problem is virtually a combinatorial optimization problem with 0-1 variables for existence of members. The problem becomes a mixed variable problem if the cross-sectional areas of the existing members are also to be optimized. In the traditional ground structure approach, optimal trusses are found by considering the cross-sectional areas as continuous design variables, and the members with vanishing cross-sectional areas are removed to obtain the optimal topology.

Let A_i and L_i denote the cross-sectional area and the length of the ith member. The number of members in the initial ground structure is denoted by m. Only inequality constraints $g_j \leq 0$ ($j = 1, 2, \cdots, n$) are considered. The truss topology optimization problem for minimizing the total structural volume is written in a form of nonlinear programming as

$$\begin{aligned} \text{P1: Minimize} \quad & V = \sum_{i=1}^{m} A_i L_i \\ \text{subject to:} \quad & g_j \leq 0, \quad (j = 1, 2, \cdots, n) \\ & A_i \geq 0, \quad (i = 1, 2, \cdots, m) \end{aligned} \tag{2.1}$$

2.2 Optimization Methods

Since topology optimization problem is formulated as a nonlinear programming problem, optimal solutions may be found by using an appropriate mathematical programming approach such as sequential quadratic programming and method of modified feasible directions. Optimality criteria methods may also be effectively used for the simple cases such as stress and/or displacement constraints for static loads. Among many methods of nonlinear programming, sequential quadratic programming is said to be most robust and accurate. Recently, new algorithms such as interior point methods (Ben-tal and Nemirovski 1994, Kočvara 1997) and semi-definite programming (Ben-Tal and Nemirovski 1996, Fujisawa *et al.* 1999) have been shown to be very effective for special cases.

In the process of removing unnecessary members, the truss may become unstable due to existence of a node without any member connecting to, or existence of a hinge that connects only two colinear members. A small lower bound $\underline{A}$ is therefore given as follows to prevent the instability:

$$A_i \geq \overline{A}, \quad (i = 1, 2, \cdots, m). \tag{2.2}$$

Note that $\overline{A}$ has a very small positive value, and the member with $A_i = \overline{A}$ should be removed from the final optimal solution; i.e. $\underline{A}$ is introduced to avoid numerical instability in the optimization process, and not to avoid unstable optimal solution. A possibility of obtaining an unstable solution may be reduced by considering multiple loading conditions. The method based on boundary cycle to restrict the feasible solution only to the truss consists of stable units may also be useful (Nakanishi and Nakagiri 1996a,b). Including topological constraints and constraints on the shape of the triangular unit is very useful to obtain practically acceptable designs (Nakanishi and Nakagiri 1997, Ohsaki and Kato 1999).

Although an optimal solution are found by simply applying a nonlinear programming approach, it is very difficult to find the *global* optimal solution even for a simple problem with stress and/or displacement constraints. Efficiency of the solution may be evaluated by comparing the objective value with a lower bound value that can be found by solving a linear programming problem neglecting the compatibility conditions (Kirsch 1989a).

The optimal topology for the stress constraints considering single loading condition is derived as a solution of a linear programming problem with member forces as design variables (Dorn *et al.* 1964, Ringertz 1985). The solution is globally optimal if the optimal truss is statically determinate. For the case where the optimal topology is unstable, some members should be added or the unstable joints should be fixed. Ringertz (1986) presented a branch-and-bound method for finding globally optimal topology and cross-sectional areas for a problem with stress and displacement constraints. The lower bound of the objective value of the optimal solution for the current topology is calculated by neglecting the stress constraints. The lower bound may also be found by neglecting displacement constraints in stead of stress constraints (Sheu and Schmit 1972).

Minimization of compliance under single loading condition without upper bound for the cross-sectional area is considered as a min-max problem for minimizing the compliance with respect to the design variable simultaneously maximizing the total potential energy with respect to the displacements (Ben-tal and Bendsøe 1993, Bendsøe *et al.* 1994). A strictly convex dual problem is then formulated and is efficiently solved by a primal-dual approach (Beckers and Fleury 1997).

Recently, new methods such as Genetic Algorithm (GA) has been successfully applied to structural optimization problems (Jenkins 1991). Grierson and Pak (1993) applied GA to a frame topology optimization problem, where the lists of topologies and cross-sectional properties as well as the nodal locations are encoded into binary strings. The null cross-sectional area is also included in the list to enable the removal of members. Hajela and Lee (1995) presented a two-level approach to ensure the stability of the truss. They also presented a scaling approach for constrained optimization problems, where the increase of the penalty term is limited to avoid too much penalty compared with the objective value. Ohsaki (1995) applied GA for finding optimal topology considering the nodal costs, and showed that the optimal topology strongly depends on the ratio of the costs between members and nodes. He also showed that the use of topological variable in addition to the cross-sectional area leads to a rapid convergence to a solution with small number of nodes and members from a highly connected ground structure.

There are also other heuristic approaches for topology optimization. Recently, some evolutionary approaches have been presented. The term *evolutionary* is very confusing because it is used in several different meanings. A GA with evolving parameters may be called evolutionary, or just simple GA may be an evolutionary approach (Kwan 1998). The algorithm with local rules like optimality criteria, and just a growing processes are also evolutionary.

Although there are many limitations in the use of ground structure approach, it is widely used, and is a kind of standard procedure in this field. Difficulties in the ground structure approach are summarized as

1. Too many members and nodes are needed in the initial design, and addition of members and nodes is very difficult.
2. The optimal topology strongly depends on the initial design, and infinite number of nodes and members are needed if the nodal locations are also to be optimized.
3. Unrealistic optimal solutions are often obtained.

4. The truss becomes unstable if too many members are removed.

Contrary to the ground structure approach, some methods have been proposed for adding nodes and members from a simple *base truss* to generate an optimal topology (Rule 1994, McKeown 1998). The drawback of obtaining unrealistic or unstable optimal solutions may be avoided if this *growing* process is used. There is no theoretically clear criteria, however, for addition of nodes and members (Kirsch 1989b, Kirsch 1996). Modification process of topology may be written in a formal grammar (Reddy and Cagan 1995a,b, Suea *et al.* 1997). Bojczuk and Mróz (1999) used sensitivity information to determine whether the candidate topology modification is acceptable.

Simulated Annealing (SA) is also useful for topology optimization (Topping *et al.* 1996, Tagawa and Ohsaki 1999). The advantage of using SA is that both continuous and discrete variables are included without any difficulty. Tabu search may also be very effective for truss topology optimization (Bennage and Dhingra 1995). Since computational cost for analysis is very large for these stochastic or heuristic methods, an efficient method for reanalysis should be incorporated for reducing the computational cost (Kirsch 1993, Kirsch 1995, Kirsch and Liu 1995).

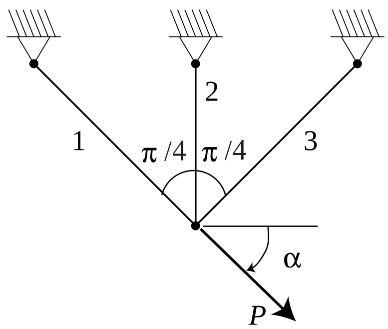


Figure 2. A three-bar truss with multiple loading conditions.

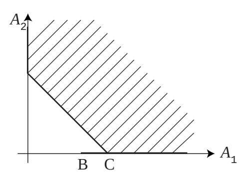


Figure 3. The feasible region of the three-bar truss.

2.3 Stress Constraints

The difficulties in topology optimization under stress constraints considering multiple loading conditions are first discussed by Sved and Ginos (1968). Detailed discussions on singularity of the solution may be referred to Kirsch (1990). Main difficulty for solving this problem with stress constraints is that the constraint need not be satisfied by a member to be removed.

Consider a three-bar truss as shown in Figure 2. (Sved and Ginos 1968). Let σ_i^k denote the stress of the ith member for the kth loading condition. The three loading conditions (P_j, α_j) = $(40, \pi/4), (30, \pi/2), (20, 3\pi/4)$ are considered. The upper and lower bounds for the stress (σ_i^L, σ_i^U) are $(-5, 5), (-20, 20), (-5, 5)$. The objective function is $V = A_2 + \sqrt{2}(A_1 + A_3)$.

The optimal solution that minimizes V under stress constraints for three loading conditions is $A_1 = 8, A_2 = 1.5, A_3 = 0$, where $V = 12.812$. Note that the stress of member 3 for P_3 is 21.4 which violates the constraint. The stress constraint, however, need not be satisfied because $A_3 = 0$. The optimal solution satisfying all the constraints is $A_1 = 7.099, A_2 = 1.849, A_3 = 2.897$, where $V = 15.986$ which is greater than the objective value 12.812 of the true optimal solution. This result suggests that the optimal solution is a singular point in the feasible region. The feasible region for this problem is illustrated in Figure 3 on the $A_1 - A_2$ plane. Note that the feasible region includes the line BC in addition to the shaded region.

Considering the discontinuity in the constraints, a ground-structure-type topology optimization problem with stress constraints is formulated as

$$\begin{aligned} \text{P2:} \quad & \text{Minimize} \quad V = \sum_{i=1}^{m} A_i L_i \\ & \text{subject to:} \quad \sigma_i^L \leq \sigma_i^k \leq \sigma_i^U, \quad \text{for } A_i > 0 \\ & \qquad\qquad (j = 1, 2, \cdots, n; \; k = 1, 2, \cdots, f) \\ & \qquad A_i \geq 0, \quad (i = 1, 2, \cdots, m) \end{aligned} \tag{2.3}$$

where f is the number of loading conditions. The formulation above indicates that the stress constraints should be relaxed in the vicinity of $A_i = 0$. Cheng and Guo (1997) presented the $\varepsilon-$relaxation method for obtaining the optimal topology by successively solving the relaxed problems. In their method, the stress constraints are relaxed as

$$(\sigma_i^L - \sigma_i^k)A_i \leq \varepsilon \tag{2.4a}$$
$$(\sigma_i^k - \sigma_i^U)A_i \leq \varepsilon \tag{2.4b}$$
$$A_i \geq \varepsilon^2 \tag{2.4c}$$

where ε is sufficiently small. The optimal topology may be found by reducing ε to 0.

Note that the stress σ_i^k of a removed member cannot be calculated from the axial force N_i^k as

$$\sigma_i^k = \frac{N_i^k}{A_i} \tag{2.5}$$

because $N_i^k = A_i = 0$. The strain e_i^k, however, can be calculated from the axial deformation d_i^k, which is easily found from the nodal displacements, as

$$e_i^k = \frac{d_i^k}{L_i} \tag{2.6}$$

and the stress may be defined by

$$\sigma_i^k = E e_i^k e \tag{2.7}$$

where E is the elastic modulus (Cheng and Jiang 1992). Therefore, there is no discontinuity in σ_i itself at $A_i = 0$ if the truss after removal of the ith member is stable, and if the definition Equation (2.5) is always used in stead of Equation (2.4); i.e. only the definition of the stress constraints is discontinuous.

For pin-jointed trusses, the lower bound σ_i^L may be replaced by the Euler buckling stress if the local buckling is considered (Cheng 1995, Smith 1997). The optimal topology strongly depends on existence of local buckling constraints and on the limits of the stresses in tension and compression especially for bridge-type structures (Achtziger 1996, Oberndorfer *et al.* 1996).

Existence of slender members can be avoided by including local buckling constraints. Global instability, however, cannot be considered by this formulation. The instability due to existence of unstable hinges between two colinear members may be avoided by fixing the hinge to obtain a truss with long members. In this case, however, the local buckling stress changes in accordance with the change in the member length and the slenderness ratio, and the constraints on local buckling might be violated. One possible solution to avoid the existence of unstable optimal topology may be to include constraints on global buckling explicitly in the problem formulation (Rozvany 1996, Zou 1996). In this case, however, unrealistic very slender members exist in the optimal topology, which has been pointed out by Nakamura and Ohsaki (1992) for a problem with frequency constraints. The possibility of obtaining a truss with long members may be reduced by introducing constraints on slenderness ratio (Achtziger 1999a,b).

2.4 Frequency Constraints

There are very few papers on topology optimization of trusses for specified fundamental frequency due to difficulties arising from local instability and multiplicity of the lowest frequency (eigenvalue). Let $\mathbf{M}_s$ and $\mathbf{M}_0$ denote the mass matrices due to structural and nonstructural masses, respectively. The eigenvalue problem of vibration is formulated as

$$\mathbf{K}\boldsymbol{\Phi}_r = \Omega_r(\mathbf{M}_s + \mathbf{M}_0)\boldsymbol{\Phi}_r, \quad (r = 1, 2, \cdots, n) \tag{2.8}$$

where Ω_r and $\boldsymbol{\Phi}_r$ are the r$^{\text{th}}$ eigenvalue and eigenvector, respectively, and n is the number of freedom of displacements. The eigenvector $\boldsymbol{\Phi}_r$ is normalized by

$$\boldsymbol{\Phi}_r^T(\mathbf{M}_s + \mathbf{M}_0)\boldsymbol{\Phi}_r = 1, \quad (r = 1, 2, \cdots, n) \tag{2.9}$$

Let $\bar{\Omega}$ denote the specified lower bound of the eigenvalues. The topology optimization problem for specified fundamental eigenvalue is formulated as

$$\begin{aligned} \text{P3:} \quad & \text{Minimize} \quad V = \textstyle\sum_{i=1}^{m} A_i L_i \\ & \text{subject to:} \quad \Omega_r \geq \bar{\Omega} \ (r = 1, 2, \cdots, n), \\ & \qquad\qquad\qquad A_i \geq 0, \quad (i = 1, 2, \cdots, m) \end{aligned} \tag{2.10}$$

The optimal topology is found by removing the members with $A_i = 0$ as the result of optimization. A small positive lower bound is usually given for A_i throughout the optimization process in order to prevent instability of the structure.

If the fundamental eigenvalue of the optimum design is simple, P3 may easily be solved by using a nonlinear programming or an optimality criteria approach (Venkayya and Tishler 1983, Sadek 1989), because there is no difficulty in calculating the sensitivity coefficients of Ω_1 with respect to A_i. It is well known, however, that the optimal topology often has multiple lowest eigenvalues. In this case, only directional sensitivity coefficients or subgradients can be calculated (Haug and Cea 1981). Although some formulations of sensitivity analysis of multiple eigenvalues have been presented (Haug and Choi 1982, Lund *et al.* 1994), it is not clear if those formulations can be used for topology optimization.

Nakamura and Ohsaki (1989) presented a parametric programming approach to trace a set of optimal solutions under multiple eigenvalue constraints and extended it to truss topology optimization (Ohsaki and Nakamura 1992). It has also been shown that their method can be applied to topology optimization of frames (Ohsaki and Nakamura 1993).

Since components of $\mathbf{K}$ and $\mathbf{M}_s$ are proportional to A_i, those are written as

$$\mathbf{K} = \sum_{i=1}^{m} A_i \mathbf{K}_i, \quad \mathbf{M}_s = \sum_{i=1}^{m} A_i \mathbf{M}_i. \tag{2.11}$$

By using the Rayleigh's principle, the constraint of Equation (2.10) is converted as

$$\sum_{i=1}^{m} (\mathbf{K}_i - \bar{\Omega} \mathbf{M}_i) A_i - \bar{\Omega} \mathbf{M}_0 \succeq \mathbf{O} \tag{2.12}$$

which means the matrix in the left-hand side should be positive semidefinite. Fujisawa *et al.* (1999) presented a method based on Semi-Definite Programming (SDP) (Kojima *et al.* 1997), and showed an optimal topology with five-fold fundamental frequencies can be found without any trouble, because SDP does not need sensitivity information of eigenvalues.

Figure 4. A simple three-bar truss with unstable joint.

Another difficulty in frequency constraints is the existence of local modes due to instability at the joints (Nakamura and Ohsaki 1992). Consider a simple three-bar truss as shown in Figure 4. If the nonstructural mass at node 3 is sufficiently large compared with the total structural mass, the lowest eigenmode at the initial design with uniform cross-sectional areas is such that the nodes 2 and 3 moves horizontally, and the axial deformation of member 3 is negligibly small compared with those of members 1 and 2. This type of mode associated with vibration of nonstructural masses is referred to as *global mode*. Since the axial deformation of member 3 is negligibly small, A_3 decreases as the optimization process proceeds. It is obvious, however, that the pin-jointed truss is unstable if member 3 is removed. Consequently, there exists a secondary member with extremely small cross-sectional area and two lowest eigenvalues coincide in this simple truss. In this case, one of the fundamental eigenmode is such that node 3 vibrates vertically and the vibration of nonstructural mass is negligibly small. This type of mode is referred to as *local mode* in the following. From the practical point of view, however, the optimal topology with secondary members are not needed, and the designers are not interested in the local mode which is simply suppressed by adding flexural stiffness at the joints.

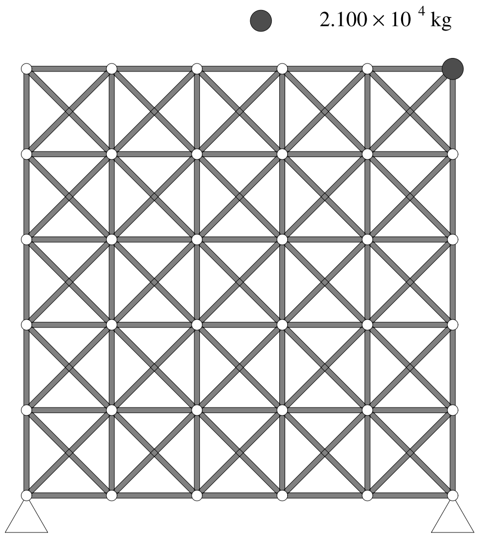


Figure 5. A 5 × 5 plane square grid.

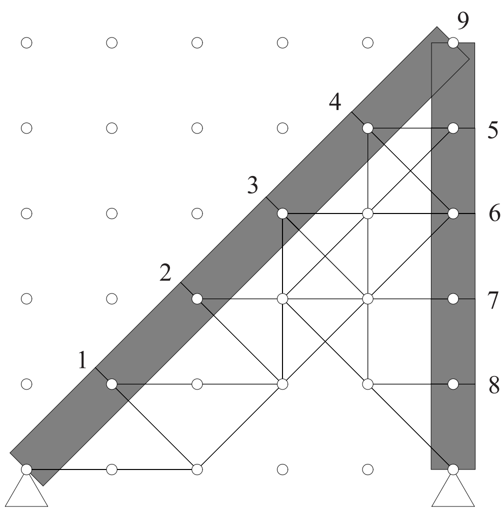
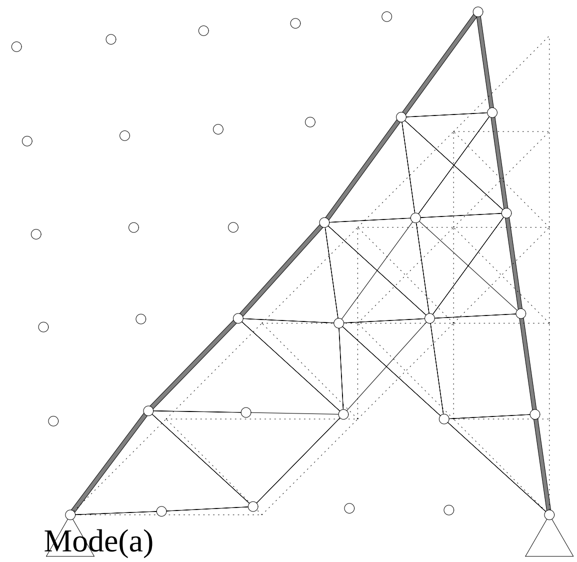


Figure 6. Optimal topology of 5 × 5 grid.

The optimal topology of a 5 × 5 plane square grid as shown in Figure 5 after removing extremely slender members is as shown in Figure 6. Note from Figure 6 that there exists a kind of net with secondary members for preventing instability of the ten-bar truss formed by the primal members with moderately large cross-sectional areas. Those secondary members cannot be removed because the two long members, each composed of five short members, will be unstable without those members. The multiplicity of the lowest eigenvalue is two, and the corresponding modes are as illustrated in Figure 7. It may be observed from Figure 7 that the displacements of node 9 where the nonstructural mass is located is very large in the mode (a), whereas local flexural deformation at node 1 dominates in the mode (b). A practically optimal topology may be found, if necessary, by removing the secondary members and fixing the unstable nodes 1-8 in Figure 5 to generate a frame with two members. Note that the node 9 is not fixed.



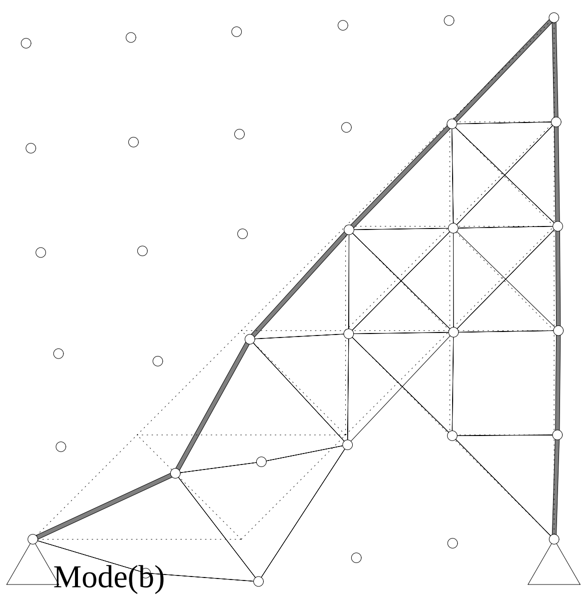


Figure 7. Eigenmodes of optimal 5 × 5 grid.

2.5 Simultaneous Optimization of Topology and Geometry

Geometries or nodal locations of trusses are easily optimized by solving a nonlinear programming considering the nodal coordinates as continuous design variables (Imai and Schmit 1982, Lin *et al.*, Dems and Gatkowski 1995). Usually side constraints or linking on design variables are introduced for avoiding intersection of members (Saka 1980, Ohkubo and Asai, 1992, Sadek 1986). Therefore, topology cannot be optimized by simply varying the nodal coordinates. Since unrealistic shape may be obtained if all the nodal coordinates are considered as independent design variables, it is useful to use parametric forms of curves and surfaces (Ohsaki *et al.* 1997, Ohsaki *et al.* 1998).

Simultaneous optimization of topology and geometry is rather easy if the members can have arbitrary positive cross-sectional areas. Rozvany presented series of works for layout optimization based on analytical optimality criteria (Zou and Rozvany 1991). Applicability of his method, however, depends on the types of constraints. His method also needs thousands of members to optimize the geometry of trusses. Rajeev and Krishnamoorthy (1997) presented a GA-based method for optimizing the nodal locations and cross-sectional areas of trusses of candidate topologies.

For the trusses where the selections of the member cross-sectional areas are limited, the problem becomes very difficult. The difficulties are summarized as (Ohsaki 1997)

1. Existence of extremely short members or elimination of some members leads to singularity of the stiffness matrix.
2. A member which has larger cross-sectional area is generated if coalescent members are simply combined into single member. The global stiffness changes discontinuously, on the contrary, if some among the coalescent members, leaving only one member, are removed.
3. Additional side constraints are needed to restrict the relative displacement of the closely spaced nodes; i.e. the relative displacements between two nodes is not a continuous function of the distance of the nodes.
4. Any node or member cannot be removed completely during the optimization process, because once-removed node or member may turn out to be necessary to exist in the final optimal solution.

The most simple approach for this problem is to find optimal geometries for several possible topologies (Shiraishi *et al.* 1981). The topology and geometry may be successively optimized by a two-level approach (Bendsøe *et al.* 1994). Ohsaki (1997) presented a continuous topology transition model for a regular plane truss, and extended it to general plane trusses (Tagawa and Ohsaki 1999).

3. CONTINUUM STRUCTURAL TOPOLOGY OPTIMIZATION

Continuum structural topology optimization differs quite significantly from discrete structural optimization as described in the preceding section. The primary difference is that the structure is treated as a solid continuum of variable topology, rather than as a finite system of beam/truss elements. The continuum approach originally evolved from distributed parameter approaches to shape optimization, as for example optimization of the distribution of thicknesses in plate structures [Olhoff, 1981]. As regions of “zero plate thickness” represented “holes” in the plate structure, it was realized that distributed parameter optimization techniques were able give rise to design and describe structures of variable topology. Structural topology optimization via distributed parameter optimization techniques *per se* was first proposed by Kohn and Strang [1986] and first demonstrated by Bendsoe and Kikuchi [1988].

3.1 Description of Structural Material Arrangements

In continuum structural topology optimization, the complete undeformed spatial domain $\boldsymbol{\Omega}_B$ of the structure being designed is designated. The structural domain can be decomposed into a designable region $\boldsymbol{\Omega}_D$ and a non-designable region $\boldsymbol{\Omega}_N$ in which the structural design is prescribed. The layout or arrangement of a pre-selected structural material in $\boldsymbol{\Omega}_D$ remains to be determined and so this region is called *designable*. A set of single or multiple loading/boundary conditions to which $\boldsymbol{\Omega}_B$ will be subjected are specified and a starting design $\mathbf{b}^{(0)}$ which specifies the initial material layout in the $\boldsymbol{\Omega}_D$ is selected. For each set of loading/boundary conditions, the performance of the structure is analyzed, with quasi-static behaviors, dynamic, and free-vibrational behaviours all treatable. For the sake of simplicity in explanation, attention will first be restricted to quasi-static single loading conditions. The objective of the design process is to iteratively improve upon the initial arrangement of structural material in $\boldsymbol{\Omega}_D$ until an optimum design with the desired performance and "cost"† characteristics is achieved. Accordingly, an objective functional which measures the mechanical performance characteristics of interest in the structure must be specified, along with constraint functionals and side constraints on the design variables.

A system is clearly needed to describe the the spatial distribution of structural material throughout the spatially fixed structural domain $\boldsymbol{\Omega}_D$. In designing the shape/topology of a structural system that uses only one structural material $\mathcal{A}$, the binary indicator function describing the arrangement of the material would be:

$$\chi_{\mathcal{A}}(\mathbf{X}) = \left\{ \begin{array}{ll} 1 & \text{if material } \mathcal{A} \text{ fully occupies point } \mathbf{X} \in \boldsymbol{\Omega}_D \\ 0 & \text{otherwise} \end{array} \right\}, \tag{3.1}$$

One can think of regions not occupied the structural material $\mathcal{A}$ as being occupied by a fictitious void material $\mathcal{B}$ whose indicator function would be:

$$\chi_{\mathcal{B}}(\mathbf{X}) = \left\{ \begin{array}{ll} 1 & \text{if material } \mathcal{B} \text{ fully occupies point } \mathbf{X} \in \boldsymbol{\Omega}_D \\ 0 & \text{otherwise} \end{array} \right\}. \tag{3.2}$$

The respective domains $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ occupied by materials $\mathcal{A}$ and $\mathcal{B}$ would simply be:

$$\Omega_{\mathcal{A}} = \{\mathbf{X} \in \boldsymbol{\Omega}_D \mid \chi_{\mathcal{A}}(\mathbf{X}) = 1; \chi_{\mathcal{B}}(\mathbf{X}) = 0; \} \tag{3.3a}$$

$$\Omega_{\mathcal{B}} = \{\mathbf{X} \in \boldsymbol{\Omega}_D \mid \chi_{\mathcal{B}}(\mathbf{X}) = 1; \chi_{\mathcal{A}}(\mathbf{X}) = 0; \} \tag{3.3b}$$

It is generally preferred to obtain final material distributions $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ that satisfy Equations (3.3). Such distributions are achieved, however, using continuous formulations which permit various forms of <u>mixtures</u> to exist throughout the design domain $\boldsymbol{\Omega}_D$ in intermediate, and even final states. By permitting mixtures, the material phases $\mathcal{A}$ and $\mathcal{B}$ are allowed to simultaneously and partially occupy an infinitesimal neighborhood about each spatial point $\mathbf{X}$ in $\boldsymbol{\Omega}_D$. In describing the mixtures, the binary indicator functions above are no longer useful, but a variety of straightforward and continuous generalizations of the binary indicator function concept are available. In fact, some of the primary differences among the myriad of continuum structural topology optimization formulations that have been presented in the literature result from the treatment of local mixtures.

The purpose of mixture descriptions is to provide mechanical performance characteristics (such as elastic stiffness) to local mixtures in $\boldsymbol{\Omega}_D$. In a broad sense, two general methods for treatment of material mixtures have been employed in continuum topology optimization:

† In continuum topology optimization, "cost" is typically associated with the volume or mass of structural material used.

(1) so-called relaxed formulations in which the arrangement of structural material in a local mixture has a definite micro-structure which can be described and quantified through a number of morphologic parameters which will also serve as the distributed parameter design variables. The effective mechanical properities of the local mixture can then be obtained using a variety of homogenization and/or micro-mechanics techniques.

(2) heuristic mixing rules which assume and involve no microstructure.

Characteristic of the relaxed formulations utilizing an assumed micro-morphology is the structured porous solid approach proposed by Bendsoe and Kikuchi [4] and widely used in structural topology design to find optimal distributions of solid and void phases. In this method, local mixtures take the form of an assumed periodic porous medium with morphology parameters (a, b, θ) which represent, respectively, the normalized dimensions of the rectangular pores and the pore orientation. Computational homogenization is employed to calculate the effective elastic constants for a discrete sequence of morphology parameters and then in topology optimization numerical interpolation is employed to compute effective elasticity properties of solid-void mixtures for any intermediate values of the morphology parameters.

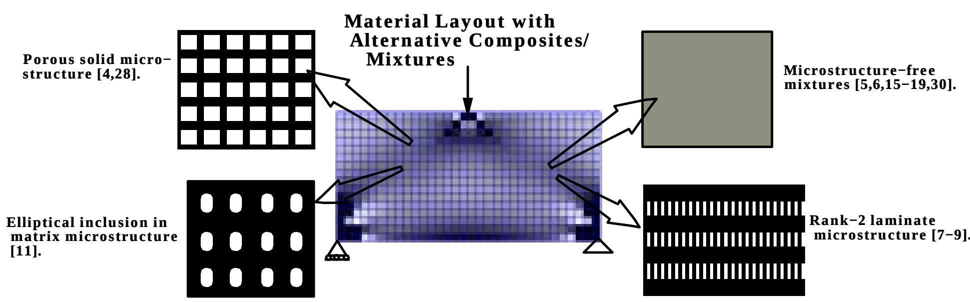


Figure 8. Schematic of variable topology material layout design formulations using alternative classes of composites and/or amorphous mixtures.

A conceptually similar method has been used by a number of investigators (for example, Jog *et al* [7]; Bendsoe *et al* [8]; Allaire and Kohn [9]) who employ structured, rank-2 plane stress laminates in two-dimensions to mix a linear elastic solid phase and a void phase. In the formulation of Jog *et al* [7] the laminates are self-adaptive in that their orientation (θ) adjusts to the local strain field to provide a stiff local mixture or "composite" [10]. One strength of this approach is that for linear elastic and void constituents, analytical formulae rather than computational homogenization is employed to calculate the effective mechanical properties of the mixture. It is recognized, however, that as modeled the self-adaptive laminates are very stiff and do not penalize mixtures of materials in compliance minimization problems as strongly as the homogenization methods do. One consequently ends up with final material layout designs that yield high overall structural stiffness designs by making extensive usage of stiff "mixtures" or "composites." From a construction viewpoint, the designs obtained with such solid-void composites are infeasible.

Still another relaxed formulation based on an assumed micro-morphology in the material layout is the Mori-Tanaka mixing rule employed by Gea [11]. The Mori-Tanaka mixing rule [12] is based on the physical assumption of dilute suspensions of ellipsoidal particles of material $\mathcal{A}$ embedded in a matrix of material $\mathcal{B}$ and uses analytical Eshelby solutions [13] to predict the elastic properties of the mixtures based solely on the volume fraction and assumed particle shapes and orientations of the respective phases. This mixing rule can in principle be used in inelastic topology optimization since its usage for inelastic materials is already established as, for example, in the constitutive modeling of elastoplastic particulate composites [14].

The second general class of continuum topology optimization formulations utilize amorphous treatments of mixtures, in which no specific micro-structure is assumed or utilized. The class of mixing rules which assume no micro-morphology of the mixture have the advantage (when used with isotropic materials) of using only a single scalar volume fraction or density parameter to characterize the two-material mixtures in both two and three dimensions. Most prominent among the mixing rules in this category are the simple density based power laws originally investigated by Bendsoe [15] and now widely used by others [16–19], and also the classical Voigt-Reuss mixing rule approaches proposed by Swan et al [1997,1998,1999]. While these mixing rules do not necessarily have an underlying physical basis, they have been effectively and convincingly employed in structural topology optimization, and are being increasingly used due to their relative simplicity. They are also readily usable with general elastic and/or inelastic structural materials [6].

As an example, the local volume fraction of a structural material $\mathcal{A}$ at a fixed spatial point $\mathbf{X}$ in the design domain $\boldsymbol{\Omega}_D$ is denoted by $\phi_{\mathcal{A}}(\mathbf{X})$ and represents *the fraction of an infinitesimal volume element surrounding point* $\mathbf{X}$ *occupied by material* $\mathcal{A}$. The volume fraction for the void material would simply be the porosity of the mixture. Natural constraints upon the spatial volume fractions for the two-material problem are:

$$0 \leq \phi_{\mathcal{A}}(\mathbf{X}) \leq 1; \quad 0 \leq \phi_{\mathcal{B}}(\mathbf{X}) \leq 1; \quad \phi_{\mathcal{A}}(\mathbf{X}) + \phi_{\mathcal{B}}(\mathbf{X}) = 1. \tag{3.4}$$

Since the last physical constraint of (3.4) states that the solid and void volume fractions at $\mathbf{X}$ are not independent, the solid-void mixes can be defined through a single parameter, which is often called either the solid volume fraction or the local density. Simple volume fraction descriptions are simply continuous generalizations of binary indicator functions.

In most continuum topology optimization frameworks, the structure's design region $\boldsymbol{\Omega}_D$ will be discretized into NEL low-order finite elements such as bilinear plate/shell shell elements or trilinear three-dimensional continuum elements. For these low-order elements, NC design variables describing the presence and arrangement of structural material $\mathcal{A}$ in each element are taken as piecewise constant. The designable distribution of material phase $\mathcal{A}$ in $\boldsymbol{\Omega}_D$ can thus be described by a vector of NCxNEL design variables $\mathbf{b}$.

3.2 Mixing Rules

Since each finite element in the model of the design domain $\boldsymbol{\Omega}_D$ contains two material volume fractions of general elastic or inelastic solids, a critical issue that must be addressed is how these materials are combined at the finite element level, or more basically at the integration point level, to form effective stress-strain relations. Toward this end, the basic properties and characteristics of Voigt, Reuss, hybrid Voigt-Reuss, and power-law mixing rules are reviewed here.

In a strict sense, the physical motivation behind the classical Voigt and Reuss mixing rules has significance only for the very special case of one-dimensional composites or mixtures. For such mixtures, the Voigt rule assumes that the phases are arranged in parallel (Figure 9) so that when loaded axially the strain in each material will be the same. The Reuss rule assumes the phases are arranged in series so that the stress in each will be the same. The one-dimensional Voigt arrangement leads to very stiff, strong mixtures, whereas the Reuss arrangement leads to compliant and weak behaviors (Figure 9).

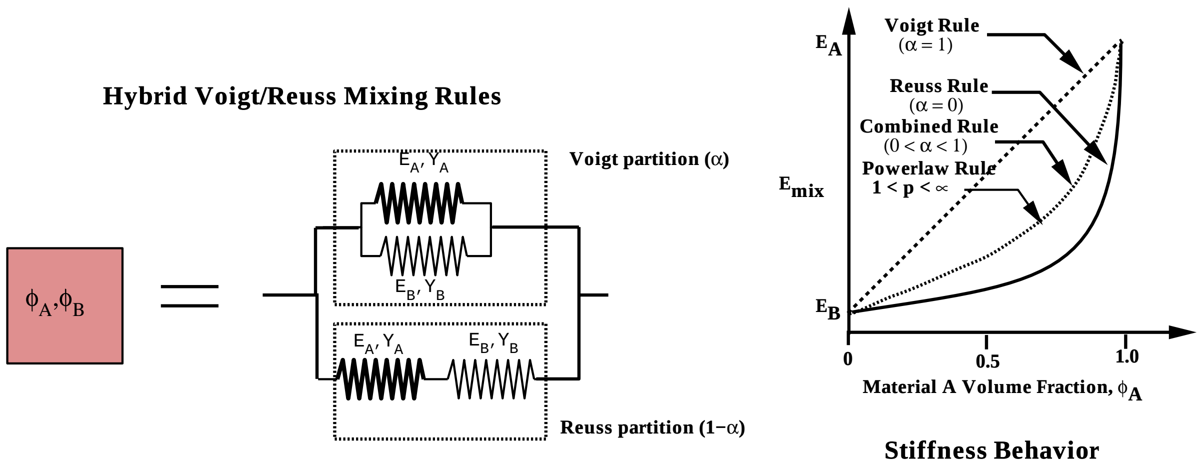


Figure 9. Schematic of effective stiffnesss of hybridized Voigt-Reuss mixtures and Power-law mixtures. For the diagrams shown, material $\mathcal{A}$ is taken to be stiffer than material $\mathcal{B}$.

For multi-dimensional mixtures, it is generally not possible to devise an equilibrium arrangement of materials that satisfies either the Voigt or Reuss conditions for all potential loading conditions [22]. Nevertheless, the Voigt and Reuss rules can be generalized to higher dimensions simply by assuming that the phases share the same local strain tensor (Voigt) or the same local stress tensor (Reuss) under all possible loading conditions. Accordingly, the decomposition equations for the Voigt mixing of two general phases at a given material point **X** are as follows:

$$\boldsymbol{\epsilon}_{\text{Voigt}} = \boldsymbol{\epsilon}_{\mathcal{A}} = \boldsymbol{\epsilon}_{\mathcal{B}} \tag{3.6a}$$
$$\boldsymbol{\sigma}_{\text{Voigt}} = \phi_{\mathcal{A}}\boldsymbol{\sigma}_{\mathcal{A}}(\boldsymbol{\epsilon}) + \phi_{\mathcal{B}}\boldsymbol{\sigma}_{\mathcal{B}}(\boldsymbol{\epsilon}). \tag{3.6b}$$

The corresponding decomposition equations for the Reuss mixing of two general phases are:

$$\boldsymbol{\epsilon}_{\text{Reuss}} = \phi_{\mathcal{A}}\boldsymbol{\epsilon}_{\mathcal{A}} + \phi_{\mathcal{B}}\boldsymbol{\epsilon}_{\mathcal{B}} \tag{3.7a}$$
$$\boldsymbol{\sigma}_{\text{Reuss}} = \boldsymbol{\sigma}_{\mathcal{A}}(\boldsymbol{\epsilon}_{\mathcal{A}}) = \boldsymbol{\sigma}_{\mathcal{B}}(\boldsymbol{\epsilon}_{\mathcal{B}}). \tag{3.7b}$$

For the hybrid Voigt-Reuss mixture (Figure 9), the assumption is that both branches of the mixture have the same strain and that the volume fraction of the total mixture in the Voigt branch is $\alpha \in [0, 1]$ and that in the Reuss branch is $1 - \alpha$. Accordingly, the effective stresses and strains of the partitioned mixture are:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\text{Voigt}} = \boldsymbol{\epsilon}_{\text{Reuss}} \tag{3.8a}$$
$$\boldsymbol{\sigma} = \alpha\boldsymbol{\sigma}_{\text{Voigt}} + (1-\alpha)\boldsymbol{\sigma}_{\text{Reuss}} \tag{3.8b}$$

Similarly, the effective stresses and strains in the powerlaw mixing rule are based on the assumption of uniform strain, but non-uniform stress:

$$\boldsymbol{\epsilon}_{\text{Power-law}} = \boldsymbol{\epsilon}_{\mathcal{A}} = \boldsymbol{\epsilon}_{\mathcal{B}} \tag{3.9a}$$
$$\boldsymbol{\sigma}_{\text{Power-law}} = \phi_{\mathcal{A}}^{p}\boldsymbol{\sigma}_{\mathcal{A}}(\boldsymbol{\epsilon}) + \phi_{\mathcal{B}}^{p}\boldsymbol{\sigma}_{\mathcal{B}}(\boldsymbol{\epsilon}) \tag{3.9b}$$

where $p \in [0, \infty)$. The limit $p = 0$ recovers the Voigt mixing rule while the limit $p \to \infty$ recovers the Reuss mixing rule.

3.3 Special Case: Linear Isotropic Solid and Void Materials

Application of the Voigt and Reuss mixing rules to the treatment of elastic and inelastic solids has been treated in [6]. Here attention is confined to the special case of mixing a linear isotropic elastic solid phase and a void phase. Young's modulus E and Poisson's ratio ν are commonly used material constants to characterize the elasticity tensor of linear isotropic elastic solids. In a mixture of a linear isotropic elastic solid and a void phase, the void phase does not contribute to the stiffness of the mixture. For simplicity, one can therefore choose the Poisson's ratio of the void phase as equal to that of the solid phase, $\nu_{\text{void}} = \nu_{\text{solid}}$. Thus, only the effective Young's modulus for the mixture E_{mix} needs to be computed since the Poisson's ratio of the mixture will be equal to that of the solid and void phases, $\nu_{\text{mix}} = \nu_{\text{void}} = \nu_{\text{solid}}$.

In the Voigt mixing rule, usage of isostrain conditions (3.6a) gives an effective Young's modulus of the mixture as:

$$E_{\text{Voigt}} = \phi_{\text{solid}} E_{\text{solid}} + \phi_{\text{void}} E_{\text{void}}. \tag{3.9}$$

Similarly, using the Reuss isostress assumption (3.7) gives an effective Young's modulus for the mixture as

$$E_{\text{Reuss}} = [\frac{\phi_{\text{solid}}}{E_{\text{solid}}} + \frac{\phi_{\text{void}}}{E_{\text{void}}}]^{-1}. \tag{3.10}$$

For mixtures containing both Voigt and Reuss partitions, the effective Young's modulus for the mixture is simply

$$E = \alpha E_{\text{Voigt}} + (1 - \alpha) E_{\text{Reuss}} \tag{3.11}$$

where α is the volume fraction of the mixture in the Voigt partition. With the Powerlaw mixing rule,

$$E = \phi^p_{\text{solid}} E_{\text{solid}} + \phi^p_{\text{void}} E_{\text{void}}. \tag{3.11}$$

Although the Young's modulus E_{void} of the void phase is theoretically zero, a small stiffness E_{void} compared to that of the solid phase E_{solid} must be maintained to avoid singularity of the finite element equations. In the event that the stiffness of the void phase is chosen too large, the structure will derive some stiffness from the void phase, which is unrealistic. Striking a balance between these two extremes, the stiffness ratio is taken as $\frac{E_{\text{void}}}{E_{\text{solid}}} = 10^{-6}$ since numerical experiments show it to provide very good performance and results.

For the special case under consideration, an advantage of mixing rule formulations is that the stiffnesses of mixtures can be written as simple, analytical, and differentiable expressions of the volume fractions $(\phi_{\text{solid}}, \phi_{\text{void}})$. As will be discussed below in the context of design sensitivity analysis, the quantity $\frac{\partial \sigma}{\partial \phi_A}|_{\mathbf{u}}$ must be computed. For the special cases under consideration, this quantity is easily computed as

$$\frac{\partial \boldsymbol{\sigma}}{\partial \phi_{\text{solid}}}|_{\mathbf{u}} = \frac{\partial \mathbf{C}}{\partial \phi_{\text{solid}}} : \boldsymbol{\epsilon}. \tag{3.12}$$

To evaluate $\frac{\partial \mathbf{C}}{\partial \phi_{\text{solid}}}$ in (3.12), one needs only to take the derivative of the effective Young's modulus E with respect to the design variables. That is,

$$\frac{\partial E_{\text{Voigt}}}{\partial \phi_{\text{solid}}} = E_{\text{solid}} - E_{\text{void}} \tag{3.13a}$$

$$\frac{\partial E_{\text{Reuss}}}{\partial \phi_{\text{solid}}} = [\frac{1}{E_{\text{void}}} - \frac{1}{E_{\text{solid}}}][\phi_{\text{solid}} E_{\text{solid}} + \phi_{\text{void}} E_{\text{void}}]^{-2} \tag{3.13b}$$

The derivative of E for hybrid mixtures is thus simply

$$\frac{\partial E}{\partial \phi_{\text{solid}}} = \alpha \frac{\partial E_{\text{Voigt}}}{\partial \phi_{\text{solid}}} + (1-\alpha)\frac{\partial E_{\text{Reuss}}}{\partial \phi_{\text{solid}}} + (E_{\text{Voigt}} - E_{\text{Reuss}})\frac{d\alpha}{d\phi_{solid}}. \tag{3.14}$$

For the Powerlaw mixtures the derivative of E is simply:

$$\frac{\partial E}{\partial \phi_{\text{solid}}} = p\phi_{\text{solid}}^{p-1} E_{\text{solid}} + p\phi_{\text{void}}^{p-1} E_{\text{void}} \tag{3.15}$$

3.4 Objective and Constraint Functionals

Numerous formulation options exist in structural topology design optimization in terms of utilizing assorted combinations of objective and constraint functionals. It is useful to distinguish between purely cost based functionals which are independent of the response of the system being designed (that is $\mathcal{F} = \mathcal{F}(\mathbf{b})$) and performance based functionals which by definition depend upon both the design variables **b** and the performance or state of the designed system which can generally be described in terms of **u**, the vector displacement field (that is $\mathcal{F} = \mathcal{F}(\mathbf{b}, \mathbf{u})$).

An example of a pure cost functional for the structural topology optimization problem is the overall volume fraction of one of the candidate constituent phases, defined as:

$$\mathcal{F}_{\phi_{\mathcal{A}}} = < \phi_{\mathcal{A}} > - \mathcal{C}_{\mathcal{A}} \tag{3.16}$$

in which $< \phi_{\mathcal{A}} >$ represents the volume average of $\phi_{\mathcal{A}}$ over the entire analysis domain $\boldsymbol{\Omega}_B$. In contrast, the global strain energy functional over the analysis domain for general loading conditions is a performance functional and would be defined as

$$\mathcal{F}_E = \int_0^t \int_B \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, d\boldsymbol{\Omega}_B \, d\tau \tag{3.17}$$

where τ is a parametric time variable. A typical topology design optimization problem might be to minimize the strain energy functional $\mathcal{F}_E$ of the structure under applied force loadings (thereby maximizing the stiffness of the structure for the applied loads), subject to a volume constraint on phase $\mathcal{A}$. Alternatively, an equally viable way to pose the problem would be to minimize the global volume fraction of material phase $\mathcal{A}$ subject to a constraint on the strain energy functional of the system. Commonly used performance functionals for linear elastic structural problems are the strain energy functional and eigenvalue functionals while examples of pure cost functionals are global volume fraction constraints; perimeter constraints [29,37]; and intermediate volume fraction penalization functionals [29,37], among many others.

The topology design optimization process is iterative in nature, requiring the solution of an analysis problem with each new variation of the design (Figure 10). Intermingled with solving the analysis problem is the evaluation of the objective and constraint functionals as well as their design gradients. The following two subsections briefly review the general linear analysis and design sensitivity analysis problems.

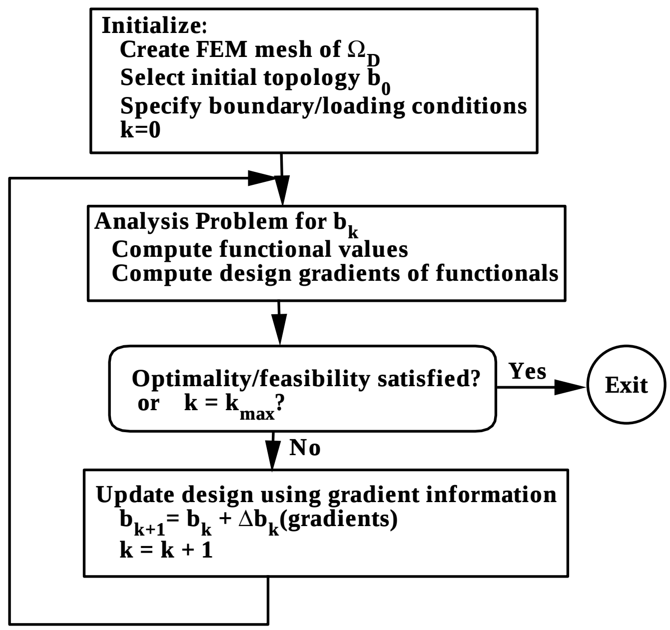


Figure 10. Topology design optimization algorithm.

3.5 Structural Analysis Problems

3.5.1 Linear Quasi-Static Analysis

Topology design can be performed to find the optimal layout of a structure to: minimize compliance; maximize strength; tailor eigenvalues; and tune dynamic response. These varied objectives require the solution of elliptic boundary value problems; eigenvalue problems; and hyperbolic initial and boundary value problems. Here, the static elliptic boundary value problems are described, and in the following subsection, vibrational eigenvalue problems are described.

If attention is restricted to the class of problems requiring solution of linear static elliptic boundary value problems, then the equilibrium state equation of the discrete structure is:

$$\mathbf{r}_A = \mathbf{f}_A^{int} - \mathbf{f}_A^{ext} = \mathbf{o} \tag{3.18}$$

where

$$\mathbf{f}_A^{int} = \int_B \mathbf{B}_A^T : \boldsymbol{\sigma}\, d\boldsymbol{\Omega}_B \doteq \mathbf{K}_{AB} \cdot \mathbf{d}_B \tag{3.19a}$$

$$\mathbf{f}_A^{ext} = \int_B \rho N_A \mathbf{f}\, d\boldsymbol{\Omega}_B + \int_{\Gamma_h} N_A \mathbf{h}\, d\Gamma_h . \tag{3.19b}$$

3.5.2 Design Sensitivity Analysis

Due to the large number of design variables, continuum structural topology optimization is typically performed using gradient based optimization techniques. Thus the analysis program must return design gradients of the objective and constraint functionals to the optimizer. For performance functionals, one must be able to compute:

$$\frac{d\mathcal{F}(\mathbf{b},\mathbf{u})}{d\mathbf{b}} = \frac{\partial \mathcal{F}}{\partial \mathbf{b}} + \frac{\partial \mathcal{F}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{b}}. \tag{3.20}$$

From (3.18) it is clear that

$$\frac{d\mathbf{r}}{d\mathbf{b}} = \mathbf{0} \tag{3.21}$$

which leads to

$$\frac{d\mathbf{u}}{d\mathbf{b}} = -\mathbf{K}^{-1}\frac{\partial \mathbf{r}}{\partial \mathbf{b}} \tag{3.22}$$

which can be inserted into (3.20) to yield

$$\frac{d\mathcal{F}}{d\mathbf{b}} = \frac{\partial \mathcal{F}}{\partial \mathbf{b}} - \mathbf{u}^a \cdot \frac{\partial \mathbf{r}}{\partial \mathbf{b}} \tag{3.23}$$

where $\mathbf{u}^a$, the “adjoint displacement vector”, is the solution of the “adjoint problem”

$$\mathbf{K} \cdot \mathbf{u}^a = -\frac{\partial \mathcal{F}}{\partial \mathbf{u}}. \tag{3.24}$$

In many topology design optimization problems, the objective is to minimize the compliance (or maximize the stiffness) of a structure to which set of fixed external loads $\mathbf{f}^{ext}$ are being applied. Minimization of the strain energy or compliance functional (4.2) associated with a prescribed loading maximizes the stiffness of the structure under the prescribed loading condition. With attention restricted to linear elastic structures, the strain energy functional reduces to the especially simple form:

$$\mathcal{F}_E = \frac{1}{2}\mathbf{f}^{ext} \cdot \mathbf{u}. \tag{3.25}$$

The displacement field that solves the equilibrium condition (3.18) for linear elastic structures is simply $\mathbf{u} = \mathbf{K}^{-1} \cdot \mathbf{f}^{ext}$, and the corresponding adjoint displacement vector $\mathbf{u}^a$ that solves the adjoint problem (3.24) is merely

$$\mathbf{u}^a = -\frac{1}{2}\mathbf{K}^{-1} \cdot \mathbf{f}^{ext} = -\frac{1}{2}\mathbf{u}. \tag{3.26}$$

As it was assumed that the external loads $\mathbf{f}^{ext}$ applied to the structure are independent of the design variables $\mathbf{b}$, the design gradient of the strain energy functional reduces to

$$\frac{d\mathcal{F}_E}{d\mathbf{b}} = -\frac{1}{2}\int_B \boldsymbol{\epsilon} : \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{b}} d\boldsymbol{\Omega}_B. \tag{3.27}$$

The quantity $\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{b}}$ in (3.27) is here termed the “stress design gradient,” and clearly depends upon the mixing rule being employed. The evaluation of this “stress design gradient” for all of the common mixing rules was described in Section 3.2.

3.5.3 Uniqueness of Solutions

When used with filtering methods[Sigmund and Diaz, 1998], stable, checkerboard-free material layout designs can be obtained. An additional issue that arises is the uniqueness of solutions obtained for fixed mesh discretizations of the domain $\boldsymbol{\Omega}_B$. One can generally assume the existence of locally optimal solutions $\mathbf{b}^*$ such that

$$\mathcal{F}_E(\mathbf{b}^*) \leq \mathcal{F}_E(\mathbf{b}^* + \delta\mathbf{b}) \tag{3.28}$$

for all feasible infinitesimal variations $\delta\mathbf{b}$ of the design vector. Most continuum topology optimization formulations when applied to linear elastic structural problems produce stable, locally optimal solutions, and thus existence does not appear to be a major issue. † In fact, the problem encountered with highly penalized formulations is that an excessive number of stable local minima satisfying (3.28) exist for a given meshing of $\boldsymbol{\Omega}_B$. The general aim in continuum topology optimization is to arrive at the discrete, interpretable, and constructable solutions whose performance characteristics approach that of the mathematical global minimum solution $\mathbf{b}_g^*$ but which is invariably neither discrete, interpretable nor manufacturable.

It was shown in [39] that usage of the stiff Voigt-like formulation leads to a uniformly convex strain energy functional $\mathcal{F}_E$, and hence unique solutions of the optimization problem for a fixed mesh discretization of $\boldsymbol{\Omega}_B$, irrespective of the starting point $\mathbf{b}_o$ or the optimization algorithm employed. Usage of highly penalized formulations, however, does not necessarily lead to a uniformly convex functional, and so the solution obtained will depend upon both the starting design $\mathbf{b}_0$ and possibly the optimization algorithm employed. While the unpenalized Voigt solutions to topology optimization problems may be unique, they are generally not very desirable in that they tend to be very grey, containing large regions of mixed materials which can be very difficult to interpret. Solutions obtained with compliant mixing rules, on the other hand, while highly discrete, will generally not be unique. That is, many locally optimal solutions may exist, some of which have performance characteristics approaching those of the global minimum, and some of which do not.

Given the strengths and deficiences of both extremes, it is desirable to combine the Voigt and Reuss formulations using hybrid mixing rules to obtain the best features of both: the uniqueness of the Voigt solutions, and the interpretability/manufacturability of highly penalized formulations. This is presently a very active research topic [37]. One heuristic way that this has been attempted [17,32] is through continuation methods in which the topology optimization problem is begun with a stiff mixing rule, and gradually transitioned to a penalized formulation. The objective behind the procedure is to get and keep the layout design in the convergence basin of the global optimum while gradually transitioning to a penalized formulation that will yield a discrete and manufacturable solution. The results shown in Section 6 and those in [17,32] demonstrate that this approach invariably gives, interpretable and manufacturable designs, and the performance (stiffness) of these designs often, but not always, exceeds the performance achieved by beginning with pure Reuss formulations. This behavior is highly problem dependent and requires further study.

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† Here, attention is restricted to the nature of solutions for fixed mesh resolutions of $\boldsymbol{\Omega}_B$. To address the broader issue of the existence of mesh-independent solutions, fixed length scales are imposed on designs either through perimeter control methods [29] or fixed length scale spatial filters [17].

3.6 Frequency Response Characteristics

Free vibrational modes of linear elastic structures are characterized by the eigenvalue equation

$$\mathbf{0} = (\mathbf{K} - \lambda_l \mathbf{M})\mathbf{y}_l, \quad \text{where} \tag{3.29}$$

where **K** is the structural stiffness matrix, **M** the mass matrix, λ_l the l^{th} vibrational eigenvalue, and $\mathbf{y}_l$ the corresponding eigenvector which describes the l^{th} mode of vibration. Since each element of $\boldsymbol{\Omega}_s$ generally contains a mixture of materials, the elastic constitutive tensor **C** at the element level used in computing **K** is provided by the mixing rules described above, while the local density of the mixture in each element used in computing **M** is given by the relation

$$\rho = \phi_{\mathcal{A}}\rho_{\mathcal{A}} + \phi_{\mathcal{B}}\rho_{\mathcal{B}}. \tag{3.30}$$

For solid-void structural applications, the density of the void phase ρ_{void} is theoretically zero, but a small density compared to that of the solid phase ρ_{solid} is maintained to avoid singularity of the eigenvalue equation (3.29).

One method for optimizing the overall stiffness of a structure without reference to any specific loadings is to maximize a functional $\mathcal{L}$ which is a linear combination of the first K vibrational eigenvalues:

$$\mathcal{L} = \sum_{k=1}^{K} \beta_k \lambda_k \tag{3.31}$$

where the λ's are vibrational eigenvalues and the β's are non-negative constant weighting factors. Other investigators studying application of continuum structural topology optimization to free vibration responses of structures have proposed a range of alternative objective and/or constraint functionals [32,40,43].

When $\mathcal{L}$ contains only simple nonrepeated eigenvalues, first order design sensitivity analysis is quite straightforward since

$$\frac{d\mathcal{L}}{d\mathbf{b}} = \sum_{k=1}^{K} \beta_k \frac{d\lambda_k}{d\mathbf{b}} \tag{3.32}$$

where

$$\frac{d\lambda_k}{d\mathbf{b}} = \frac{\mathbf{y}_k \cdot (\frac{\partial \mathbf{K}}{\partial \mathbf{b}} - \lambda_k \frac{\partial \mathbf{M}}{\partial \mathbf{b}}) \cdot \mathbf{y}_k}{\mathbf{y}_k \cdot \mathbf{M} \cdot \mathbf{y}_k}. \tag{2.26}$$

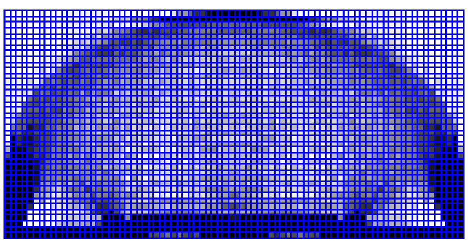
When $\mathcal{L}$ contains nonsimple, repeated eigenvalues, sensitivity analysis for the repeated roots can be somewhat more complicated. Unless other precautions are taken, the procedures suggested in [41,42] for repeated roots are usually required. However, for those classes of problems where nonsimple vibrational eigenvalues occur due strictly to the symmetry of the structure, then the symmetry reduction methods of [33] can be employed to alleviate the difficulty and design sensitivity analysis of functionals containing repeated eigenvalues can proceed along the lines of (3.32) and (3.33) without additional complications or precautions.

3.7 Representative Examples

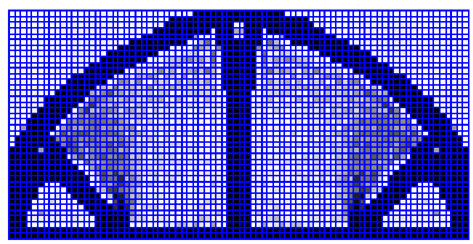
The examples that follow are representative of those that have been solved in the research literature with continuum structural topology optimization techniques. The solution algorithm employed to obtain the example solutions was a variation of a sequential linear programming algorithm (SLP) with line-searching and an LP sub-problem move limit of $\Delta_M = 0.05$. SLP methods are proving increasingly popular for large-scale topology optimization applications [19,32].

3.7.1 The Three-Load Bridge Problem

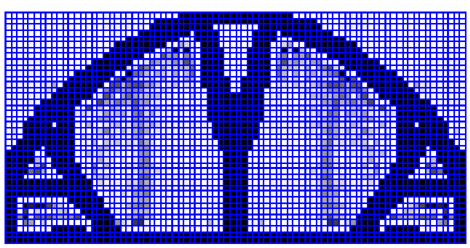
The design domain Ω_D shown in Figure 11a is originally completely filled with a linear isotropic solid material of properties $\lambda_{\text{solid}} = 2.69 \cdot 10^5$ and $\mu_{\text{solid}} = 1.15 \cdot 10^5$, and the loading and restraint conditions on the domain are as shown in Figure 11. This is a solid material–void material topology design problem, where the objective is to place the solid material throughout the design domain Ω_D to minimize the compliance of the structure for the loading shown, subject to a global volume fraction constraint on the solid phase of 40 percent. This problem was solved four times with varying Voigt-Reuss mixture parameters and using the filtering method proposed in [39].



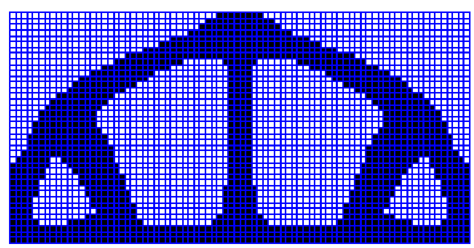
a) F_E=1.4121x10^{-2}; α=1.0



b) F_E=2.0813x10^{-2}; α=0.1



c) F_E=2.3522x10^{-2}; α=0.01



d) F_E=1.6933x10^{-2}; α=0.00

Figure 11. Alternative topology designs and strain energy functional values $\mathcal{F}_E$ for varied mixing rule parameters α.

The pure Voigt solutions (Figure 11a) are extremely stiff and "grey", as one might expect due to the incompliant nature of the mixing rule, and thus lacks sharpness and clarity. Structural topology solutions obtained with the Voigt mixing rule for a broad range of test problems visually resemble those obtained using stiff rank-2 laminate mixing rules as, for example in [7,9]. The material layout designs shown with penalized formulations $\alpha = 0.1,\ .01,\ 0.0$ in Figures 11b–11d, even when some grey fringes remain, appear for the most part quite interpretable.

3.7.2 The MBB Beam Problem

Variations of this challenging test problem involving the design a European Airbus floor beam have been solved previously in [17, 29, 35]. In the variation presented here, the simply supported beam features a solid non-designable border $\boldsymbol{\Omega}_N$ with a designable interior $\boldsymbol{\Omega}_D$. This problem is solved with $\mathcal{F}_{\text{solid}} = < \phi_{\text{solid}} > -0.50 \leq 0$ and with spatial filtering. Figure 12a shows a moderately grey layout solution obtained with a hybrid Voigt-Reuss formulation (α = .05) while results from a continuation of the same design with a pure Reuss formulation (α = 0) are shown in Figure 12c.

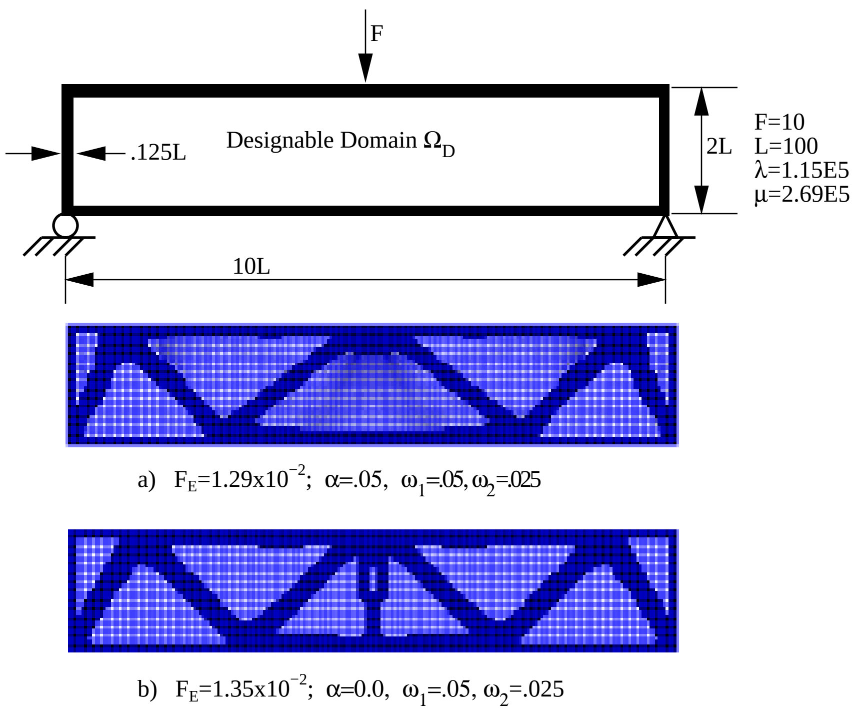


Figure 12. The MBB design problem, and solutions obtained with α = 0.05 (12a), and as a continuation of the same problem with α = 0.0 (12b).

3.7.3 The Cantilever Problem

The cantilever beam design problem (Figure 13) has been used extensively as a test problem by a number of investigators, as for example in [9,17,19,28,31,36], with the mis-aligned version of this problem considered to be especially challenging in that it tests the inherent mesh-dependency (if it exists) of alternative topology design formulations. Here, we solve this as a compliance minimization problem with a 25% global solid volume fraction constraint for both an aligned mesh, and a mis-aligned mesh (45 x 90) consisting of rectangular elements having an aspect ratio of 2:1.

Figure 13a shows the pure Reuss solution (α = 0) for the aligned mesh problem without symmetry control. Even without symmetry control, the design is symmetric, as one would expect, due to the symmetry of the mesh and the material response to the applied load. Figure 13b shows the design obtained when the same basic problem is solved in pure Reuss mode (α = 0) with a misaligned mesh of rectangular elements. In this case, there is in fact some mesh dependency of the formulation and a moderately asymmetric design is achieved. Figure 13c is a re-solving of the same mis-aligned mesh problem using continuation (α = $1.0, .01, .0001, 0$). The resulting design still shows asymmetry in the design. Finally, to show that symmetric designs can indeed be obtained even when the mesh is asymmetric with respect to the applied loads, the problem is solved once again in pure Reuss mode (Figure 13d) with gross symmetry control [33,34] imposed about the centroidal axis of the beam.

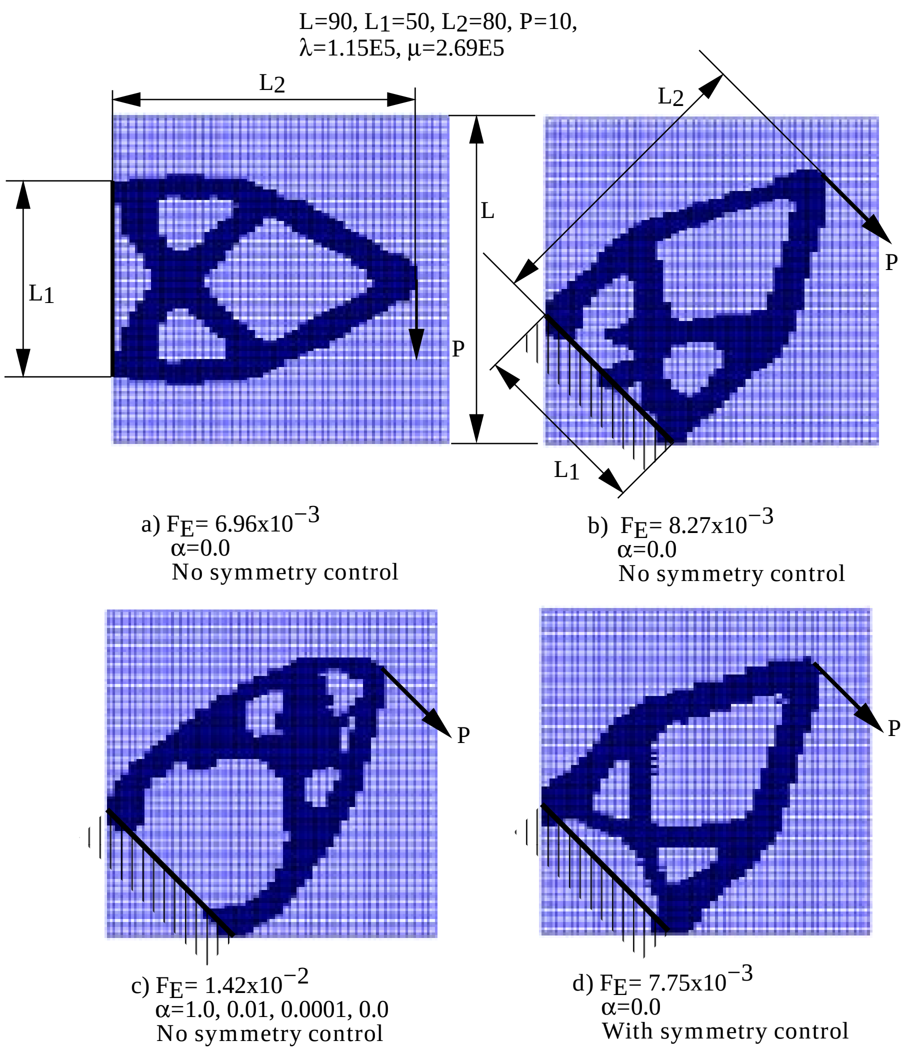


Figure 15. The cantilever design problem and solutions obtained.

3.7.4 Plate Vibration Problem

The design of material layout to maximize natural frequencies in a pin-supported square plate (Figure 14) is demonstrated here. The objective function is a weighted sum $\mathcal{L} = \sum_{k=1}^{5} \lambda_k \beta_k$ of the first five eigenvalues of the square plate, in which the λ_k and β_k are respectively the k^{th} eigenvalue and weighting factor. [Eigenvalues are related to eigenfrequences by the relation $\lambda_k = \omega_k^2$.] Since the structural domain $\boldsymbol{\Omega}_s$ is square and symmetrically restrained, the second eigenvalue is non-simple with multiplicity $s = 2$ with the modes shown in Figure 14b,c. This non-simple eigenvalue is thus treated in the objective function $\mathcal{L}$ as the second and third eigenvalues. The eigenvalue maximization problem was solved with material constraint $< \phi_{solid} > \ \leq 0.50$. Figure 14d shows the material layout solution obtained using a highly penalized mixing rule with the first eigenmode shown.

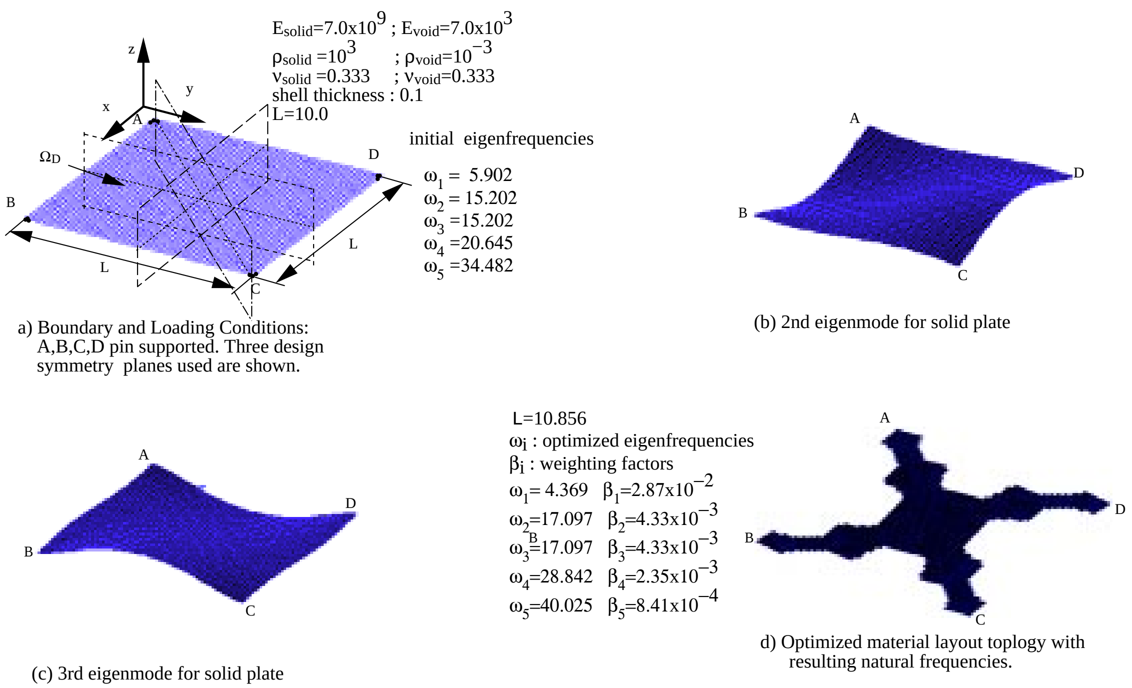


Figure 14. Square plate eigenvalue optimization problem. a) Plate properties, restraint conditions, three design symmetry planes and natural frequencies before topology optimization ; b) second eigenmode for solid plate; c) third eigenmode for solid plate; and d) optimized plate topology (first eigenmode shown) with resulting first five eigenfrequencies.

4. DISCUSSION/RECOMMENDATIONS FOR FUTURE RESEARCH

Although much literature has been published for truss topology optimization, there exist several difficulties as follows to be overcome to establish really practically efficient method.

a. Computational cost for obtaining the optimal solution for a real-world problem is still too large. More efficient algorithm should be developed using, e.g., approximation method for sensitivity analysis and reanalysis, new approaches of mathematical programming, and techniques of computer science such as parallel and distributed computing.

b. Difficulties due to local instability should be overcome to find practically admissible designs. Incorporation of practical constraints such as constraints on unit shapes may be a key idea to avoid troubles arising from seeking for theoretically correct optimal solutions.

c. Since strictly global optima is not needed in the practical use, calculating good lower and upper bound of the objective value is very helpful to determine the efficiency of the solution. An interactive approach (Smith 1996) and methods based on artificial intelligence (Hajela and Sangameshwaran, 1990) may be useful for finding a practically admissible design.

d. Many researchers believe optimal topology and geometry of trusses can be found starting from continuum-type optimal topology (Díaz and Bendsøe 1992). The characteristics of the trusses and continuum structures are very different, and there is no criteria for constructing truss topology from the results of optimal finite element solutions. Possibility of finding optimal shape of framed structures from the result of continuum type optimization process should be extensively discussed.

The authors believe the topology optimization of trusses involves difficult mathematical backgrounds and may be more difficult than continuum-type optimization problems. Collaboration of the researchers in mechanical engineering, civil engineering, computer science and applied mathematics is highly recommended toward further improvement of the optimization methods.

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